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Truncated Taylor Solvers for Simple First-Order Ordinary Differential Equations

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Abstract

Certain exact solutions of first-order ordinary differential equations naturally become corresponding solvers for simple first-order ordinary differential equations whose form is the equality of a twice continuously differentiable function of the dependent variable to the derivative with respect the independent variable of the dependent variable. When the twice continuously differentiable function of the dependent variable is replaced by its truncated Taylor expansions through second order about its initial value, the resulting first-order ordinary differential equations have exact solutions that naturally become corresponding solvers for those simple first-order ordinary differential equations.

1. Solvable Truncated Taylor Expansions of Simple First-Order Ordinary Differential Equations

We shall develop truncated Taylor solvers for simple first-order ordinary differential equations of the form,

$$dq/du = f(q), \tag{1.1}$$

where $f(q)$ is twice continuously dierentiable, and the specied initial value of $q(u)$ is $q(u_i) = q_i$. *The first three truncated Taylor expansions of the function f(q) about the specied initial value* $q_i = q(u_i)$ of $q(u)$ *produce exactly solvable approximations to the* Eq. (1.1) *exact equation* $dq = du = f(q)$, *which are,*

$$dq/du \approx f(q_i), \tag{1.2}$$

$$dq/du \approx f(q_i) + f'(q_i)(q - q_i) = f'(q_i)\big[(q - q_i) + (f(q_i)/f'(q_i))\big], \tag{1.3}$$

$$\begin{aligned} &\textit{and} \quad dq/du \approx f(q_i) + f'(q_i)(q - q_i) + \tfrac{1}{2}f''(q_i)(q - q_i)^2 = \\ &\tfrac{1}{2}f''(q_i)\big[(q - q_i)^2 + 2(f'(q_i)/f''(q_i))(q - q_i) + (2f(q_i)/f''(q_i))\big]. \end{aligned} \tag{1.4}$$

2. Exact Solutions of the Truncated Taylor Equations and the Naturally Corresponding Solvers

The exact solution for initial value $q(u_i) = q_i$ of the Eq. (1.2) truncated equation $dq/du \approx f(q_i)$ is obviously,

$$q(u) = q_i + f(q_i)(u - u_i), \tag{2.1a}$$

which naturally corresponds to the following basic solver for the Eq. (1.1) exact equation $dq = du = f(q)$,

$$q(u_{i+1}) = q(u_i) + f(q(u_i))(u_{i+1} - u_i),\ i = 0, \ldots, n. \tag{2.1b}$$

Assuming that $f'(q_i) \neq 0$, we reexpress the Eq. (1.3) truncated dierential equation as,

$$dq/\big((q - q_i) + (f(q_i)/f'(q_i))\big) = f'(q_i)\,du, \tag{2.2a}$$

which yields,

$$\ln\big((q(u)-q_i)+(f(q_i)/f'(q_i))\big)=f'(q_i)\,u+k_i, \tag{2.2b}$$

with k_i an integration constant. Since $q(u_i)=q_i$, $k_i=\ln(f(q_i)/f'(q_i))-f'(q_i)\,u_i$, so Eq. (2.2b) becomes,

$$\ln\big((q(u)-q_i)+(f(q_i)/f'(q_i))\big)=\ln(f(q_i)/f'(q_i))+f'(q_i)(u-u_i). \tag{2.2c}$$

Exponentiating both sides of Eq. (2.2c) produces,

$$(q(u)-q_i)+(f(q_i)/f'(q_i))=(f(q_i)/f'(q_i))\exp(f'(q_i)(u-u_i)), \tag{2.2d}$$

so the solution with initial value $q(u_i) = q_i$ of the Eq. (1.3) truncated dierential equation is,

$$q(u)=q_i+(f(q_i)/f'(q_i))\big[\exp(f'(q_i)(u-u_i))-1\big], \tag{2.2e}$$

which naturally corresponds to the following solver for the Eq. (1.1) exact dierential equation $dq / du = f(q)$,

$$q(u_{i+1})=q(u_i)+(f(q(u_i))/f'(q(u_i)))\big[\exp(f'(q(u_i))(u_{i+1}-u_i))-1\big],\ i=0,\ldots,n. \tag{2.2f}$$

This more sophisticated Eq. (2.2f) solver reduces to the basic Eq. (2.1b) solver in the limit $f'(q(u_i)) \to 0$.
Assuming that $f''(q_i) \neq 0$, we can reexpress the Eq. (1.4) truncated dierential equation as follows,

$$dq\big/\big((q-q_i+b_i)^2+\big(c_i-b_i^2\big)\big)=a_i du, \tag{2.3a}$$

where $a_i \stackrel{\text{def}}{=} \frac{1}{2}f''(q_i)$, $b_i \stackrel{\text{def}}{=} (f'(q_i)/f''(q_i))$ and $c_i \stackrel{\text{def}}{=} (2f(q_i)/f''(q_i))$. When $b_i^2<c_i$ the left side of Eq. (2.3a) is integrated in terms of the arctan function, but when $b_i^2>c_i$, the left side of Eq. (2.3a) is integrated in terms of the inverse of the tanh function. In the former case, integration of Eq. (2.3a) yields,

$$\Big(\arctan\Big((q(u)-q_i+b_i)\Big/\sqrt{c_i-b_i^2}\Big)\Big/\sqrt{c_i-b_i^2}\Big)=a_iu+k_i, \tag{2.3b}$$

where k_i is a constant of integration. Since $q(u_i) = q_i$,

$$k_i=-a_iu_i+\Big(\arctan\Big(b_i\Big/\sqrt{c_i-b_i^2}\Big)\Big/\sqrt{c_i-b_i^2}\Big), \tag{2.3c}$$

which when inserted into Eq. (2.3b) yields,

$$\Big(\arctan\Big((q(u)-q_i+b_i)\Big/\sqrt{c_i-b_i^2}\Big)\Big/\sqrt{c_i-b_i^2}\Big)=a_i(u-u_i)+\Big(\arctan\Big(b_i\Big/\sqrt{c_i-b_i^2}\Big)\Big/\sqrt{c_i-b_i^2}\Big). \tag{2.3d}$$

We now multiply both sides of Eq. (2.3d) by $\sqrt{c_i-b_i^2}$, followed by taking the tangent of both sides of the result, followed by multiplying both sides of that second result by $\sqrt{c_i-b_i^2}$. The upshot is,

$$q(u)=q_i-b_i+\sqrt{c_i-b_i^2}\tan\Big(a_i(u-u_i)\sqrt{c_i-b_i^2}+\arctan\Big(b_i\Big/\sqrt{c_i-b_i^2}\Big)\Big). \tag{2.3e}$$

We next apply the trigonometric identity $\tan(\theta_1+\theta_2)=(\tan\theta_1+\tan\theta_2)/(1-\tan\theta_1\tan\theta_2)$ to Eq. (2.3e), but to keep the resulting expression reasonably compact we simultaneously introduce the abbreviation,

$$W_i(u-u_i)\stackrel{\text{def}}{=}\Big(1\Big/\sqrt{c_i-b_i^2}\Big)\tan\Big(a_i(u-u_i)\sqrt{c_i-b_i^2}\Big). \tag{2.3f}$$

The upshot of the steps just described is that Eq. (2.3e) becomes,

$$q(u) =$$
$$q_i - b_i + \left(\left(\left(c_i - b_i^2\right)W_i(u - u_i) + b_i\right)\big/\left(1 - b_iW_i(u - u_i)\right)\right) =$$
$$q_i + \left(\left(-b_i\left(1 - b_iW_i(u - u_i)\right) + \left(c_i - b_i^2\right)W_i(u - u_i) + b_i\right)\big/\left(1 - b_iW_i(u - u_i)\right)\right) =$$
$$q_i + (c_iW_i(u - u_i)/(1 - b_iW_i(u - u_i))) =$$
$$q_i + \left(\left(c_i\Big/\sqrt{c_i - b_i^2}\right)\tan\left(a_i(u - u_i)\sqrt{c_i - b_i^2}\right)\Big/\left(1 - \left(b_i\Big/\sqrt{c_i - b_i^2}\right)\tan\left(a_i(u - u_i)\sqrt{c_i - b_i^2}\right)\right)\right). \quad (2.3g)$$

Eq. (2.3g) of course *is valid only if* $b_i^2 < c_i$. However, if $b_i^2 > c_i$, *the corresponding result is,*

$$q(u) =$$
$$q_i + \left(\left(c_i\Big/\sqrt{b_i^2 - c_i}\right)\tanh\left(a_i(u - u_i)\sqrt{b_i^2 - c_i}\right)\Big/\left(1 - \left(b_i\Big/\sqrt{b_i^2 - c_i}\right)\tanh\left(a_i(u - u_i)\sqrt{b_i^2 - c_i}\right)\right)\right), \quad (2.3h)$$

and if $b_i^2 = c_i$, *the corresponding result is the* $c_i \to b_i^2$ *limit of* Eq. (2.3g) *or* Eq. (2.3h), which is,

$$q(u) =$$
$$q_i + b_i(a_ib_i(u - u_i)/(1 - a_ib_i(u - u_i))). \quad (2.3i)$$

To actually utilize Eqs. (2.3i), (2.3h) *and* (2.3g) *as solvers for the* Eq. (1.1) *exact differential equation* $dq/du = f(q)$, *one must carry out extensive checks at every solving step.* At each step one first checks whether $f''(q(u_i)) = 0$. If so, one checks whether $f'(q(u_i)) = 0$. If that is also the case, one must use the basic Eq. (2.1b) solver for that particular step, but if $f'(q(u_i)) \neq 0$, one may use the Eq. (2.2f) solver for that step. If $f''(q(u_i)) \neq 0$, one computes $a(u_i) \stackrel{\text{def}}{=} \frac{1}{2}f''(q(u_i))$, $b(u_i) \stackrel{\text{def}}{=} (f'(q(u_i))/f''(q(u_i))$ and $c(u_i) \stackrel{\text{def}}{=} (2f(q(u_i))/f''(q(u_i))$ *anew for that particular step.* If $(b(u_i))^2 = c(u_i)$, one uses the Eq. (2.3i) $q(u_{i+1})$ as the solver for that step, whereas if $(b(u_i))^2 > c(u_i)$, one uses the Eq. (2.3h) $q(u_{i+1})$ as the solver for that step, but if $(b(u_i))^2 < c(u_i)$, one uses the Eq. (2.3g) $q(u_{i+1})$ as the solver for that step. *The entire procedure described in this paragraph must be repeated afresh at each and every solving step.*