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# Three-Frequency Quaternion Fourier Transform

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## 1. Introduction

As is known, the spectra of real analog signals are calculated using the Fourier Transform (FT). The FT calculated as a definite integral over time over the duration of the signal. The kernel of the transformation is the exponential to the power of a complex number $-i\omega t$, where $i$ – imaginary unit of a complex number, $\omega$ – circular frequency, $t$ – time. The exponential to the power of a complex number can be represented by Euler's formula as a sum of harmonic functions: $\exp\{-i\omega t\} = \cos(\omega t) - i\sin(\omega t)$. Integrating a signal over time allows us to calculate what portion of the signal's energy is concentrated at certain frequencies. Therefore, FT gives us a representation of the analog signal in the form of harmonic components with the corresponding amplitude and phase. The phase of harmonics is determined by the amplitudes of the cosine and sine values on the orthogonal coordinate axes of the complex plane, i.e., by the angle of the corresponding vector. The length of the vector is calculated using the Pythagorean theorem from the value of the amplitudes of the cosine and sine.

Since real signals are used in radio engineering, only the real part was extracted from the complex representation of signals. The transmission line of complex signals using both real and imaginary parts in matrix representation is considered [1]. It is shown that in this case the signal energy is the sum of the scalar and vector product. Therefore, a complex signal combines energies of different nature: potential and kinetic. Since the signal on the complex plane is defined by coordinates on two orthogonal axes, in matrix representation we obtain a MIMO 2x2 scheme. As a result, a gain of 2 times in noise immunity was achieved compared to BPSK. This result is also obtained in the Alamouti scheme [2].

A transmission line with a single-frequency quaternion is considered [3]. A quaternion in algebraic notation has the form: $q=s+ix+jy+kz$, where $i, j, k$ are imaginary units, $s, x, y, z$ are real numbers. In polar representation, a single-frequency quaternion is written as $q(\omega,t) = e^{\hat{i}\omega t} = \cos\omega t + \hat{i}\sin\omega t$, where $\hat{i} = (i+j+k)/\sqrt{3}$ is the imaginary unit of the single-frequency quaternion, $\hat{i}^2 = -1$. Imaginary units $i, j, k$ together with the real number (scalar) s form a 4D space and, accordingly, increase the separability (diversity) of signals. Since each point in 4D space is defined by 4 coordinates, in the matrix representation we obtain a MIMO 4x4 channel, which allows increasing the communication line capacity by 4 times.

In general, the quaternion FT is calculated as an integral in a 4D domain. FT of a single-frequency quaternion signal in matrix representation was considered in [4]. It is known that multiplying a quaternion by another quaternion also results in a quaternion. Therefore, the integral of a quaternion can be divided into the sum of the integrals for the scalar part and for the imaginary parts. In matrix representation, quaternion multiplication can be performed by multiplying a quaternion, in vector form, by a quaternion, in matrix form. As a result, we obtain a vector that can also be considered as a quaternion in vector representation. Therefore, the calculation of the FT of a single-frequency quaternion in matrix representation can be represented as an integral of each element of the resulting vector, and not as a volume integral in 4D. Using this representation, quaternion Fourier transforms (QFT) and corresponding inverse transforms of various impulses are obtained [4]. The properties of the shift of the elements of the quaternion vector over different time and frequency intervals are also obtained. The validity of Parseval's equality is demonstrated. Using the QFT technique for a single-frequency quaternion, similar calculations are shown for the discrete quaternion Fourier transform [5].

When solving technical problems, there are often cases where elements of some systems interact with each other, creating vibrations with different frequencies. These vibrations interact with each other to form a common frequency spectrum. When analyzing this spectrum using conventional single-frequency Fourier transforms, the core of which is complex variables, it is difficult to determine the contribution of the generating frequencies to the resulting spectrum and, accordingly, to determine the degree of influence of various elements of the system on the overall spectrum.

For example, in color image processing tasks, the entire spectrum of color images is formed using three primary colors, such as RGB (red, green, blue) or CMY (cyan, magenta, yellow) [6]. These models are associated with the peculiarities of human color perception and the presence of three types of cone receptors in the eye, sensitive to three primary colors.

Three primary colors with a positive sign RGB and a negative sign CMY form the so-called color cube. Each point of the cube represents a certain color, obtained by adding together in 3D space the constituent colors, i.e., spatial frequencies, on the axes.

A communication line with a three-frequency quaternion is considered [7]. It is shown that the use of three different angular frequencies for each imaginary coordinate axis of 4D space increases the possibilities of using the frequency resource to increase throughput.

The purpose of this paper is to present a method for the quaternion Fourier transform of a three-frequency quaternion.

## 2. Materials and Methods for Solving the Problem

The use of a three-frequency quaternion for the Fourier transform causes certain difficulties, since the exponential of such a quaternion is the product of the cosines and sines of different angular frequencies $\omega_i$, $\omega_j$, $\omega_k$ on the axes *i, j, k* [7]:

$$\mathrm{e}^q = \exp\left\{s + i\omega_i t + j\omega_j t + k\omega_k t\right\} = \tag{1}$$

$$= \mathrm{e}^s \left(\cos\omega_i t + i\sin\omega_i t\right)\left(\cos\omega_j t + j\sin\omega_j t\right)\left(\cos\omega_k t + k\sin\omega_k t\right).$$

After multiplying the expressions in parentheses in formula (1) and grouping by real and imaginary parts, we obtain an exponential function in the form of a three-frequency quaternion, which we denote as

$$f\left(q(\omega_i,\omega_j,\omega_k,t)\right) = \tag{2}$$

$$= p(\omega_i,\omega_j,\omega_k,t) + iu(\omega_i,\omega_j,\omega_k,t) + jv(\omega_i,\omega_j,\omega_k,t) + kw(\omega_i,\omega_j,\omega_k,t).$$

After reducing similar terms, the components in expression (2) will take the form:

$$p(\omega_i,\omega_j,\omega_k,t) = \cos\omega_i t\cos\omega_j t\cos\omega_k t - \sin\omega_i t\sin\omega_j t\sin\omega_k t, \tag{3}$$

$$u(\omega_i,\omega_j,\omega_k,t) = \sin\omega_i t\cos\omega_j t\cos\omega_k t + \cos\omega_i t\sin\omega_j t\sin\omega_k t,$$

$$v(\omega_i,\omega_j,\omega_k,t) = \cos\omega_i t\sin\omega_j t\cos\omega_k t - \sin\omega_i t\cos\omega_j t\sin\omega_k t,$$

$$w(\omega_i,\omega_j,\omega_k,t) = \sin\omega_i t\sin\omega_j t\cos\omega_k t + \cos\omega_i t\cos\omega_j t\sin\omega_k t.$$

In matrix representation we obtain the fundamental matrix:

$$\mathbf{\Phi}(\omega_i,\omega_j,\omega_k,t) = \tag{4}$$

$$= p(\omega_i,\omega_j,\omega_k,t)\mathbf{E} + u(\omega_i,\omega_j,\omega_k,t)\mathbf{I} + v(\omega_i,\omega_j,\omega_k,t)\mathbf{J} + w(\omega_i,\omega_j,\omega_k,t)\mathbf{K},$$

where (5)

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{I} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathbf{J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} -$$

*quaternion basis matrices.*

Typically, QFT is defined formally as an integral over a 4D volume [8,9]. However, no calculations of signal spectra are shown. It is obvious that calculating the integral over volume causes significant difficulties.

A QFT technique is known for a single-frequency quaternion in a matrix representation, in which the integral over a 4D volume is presented in the form of 4 one-dimensional integrals for each orthogonal spatial axis *s, i, j, k* and the results of calculating the QFT of various 4D pulses are shown [4].

As can be seen from expressions (2), (3), (4), each element of the fundamental matrix (4) depends on 3 angular frequencies. Therefore, the calculation of 3-frequency QFT (3fQFT) must be performed for each frequency while fixing the values of the other 2 frequencies. Obviously, such a task is also complicated and does not allow obtaining a result in the form of a formula.

It is shown that it is possible to decompose the fundamental matrix (4) for three reference angular frequencies $\omega_i$, $\omega_j$, $\omega_k$ into combination frequencies $\Omega_n$, $n = 1,2,3,4$, [7]. Positive combination frequencies are calculated as

$$\Omega_1 = \omega_i + \omega_j + \omega_k\,,\ \Omega_2 = \omega_i + \omega_j - \omega_k\,,\ \Omega_3 = \omega_i - \omega_j + \omega_k\,,\ \Omega_4 = \omega_i - \omega_j - \omega_k\,. \tag{6}$$

The matrix for transforming reference frequencies into combined frequencies is a pseudo-inverse:

$$\mathbf{\Omega} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}. \tag{7}$$

Using $\mathbf{\Omega}^{\mathrm{T}}$, the vector of reference frequencies $\boldsymbol{\omega} = \begin{bmatrix} \omega_i & \omega_j & \omega_k \end{bmatrix}^{\mathrm{T}}$ is calculated with a known vector $\mathbf{\Omega} = \begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 \end{bmatrix}^{\mathrm{T}}$ of combination frequencies (6).

Using known trigonometry formulas for products of three combinations with sines and cosines, expressions for the components of the three-frequency quaternion (3) were obtained through positive combination frequencies [17]:

$$\begin{aligned} &4p(\Omega_1,\Omega_2,\Omega_3,\Omega_4,t) = \\ &= \cos(\Omega_1 t) + \sin(\Omega_1 t) + \cos(\Omega_2 t) - \sin(\Omega_2 t) + \cos(\Omega_3 t) - \sin(\Omega_3 t) + \cos(\Omega_4 t) + \sin(\Omega_4 t)\,, \\ &4u(\Omega_1,\Omega_2,\Omega_3,\Omega_4,t) = \\ &= -\cos(\Omega_1 t) + \sin(\Omega_1 t) + \cos(\Omega_2 t) + \sin(\Omega_2 t) + \cos(\Omega_3 t) + \sin(\Omega_3 t) - \cos(\Omega_4 t) + \sin(\Omega_4 t)\,, \\ &4v(\Omega_1,\Omega_2,\Omega_3,\Omega_4,t) = \\ &= \cos(\Omega_1 t) + \sin(\Omega_1 t) - \cos(\Omega_2 t) + \sin(\Omega_2 t) + \cos(\Omega_3 t) - \sin(\Omega_3 t) - \cos(\Omega_4 t) - \sin(\Omega_4 t)\,, \\ &4w(\Omega_1,\Omega_2,\Omega_3,\Omega_4,t) = \\ &= -\cos(\Omega_1 t) + \sin(\Omega_1 t) - \sin(\Omega_2 t) - \cos(\Omega_2 t) + \sin(\Omega_3 t) + \cos(\Omega_3 t) + \cos(\Omega_4 t) - \sin(\Omega_4 t)\,. \end{aligned} \tag{8}$$

Using expressions (8), the fundamental matrix (4) for reference frequencies $\omega_i$, $\omega_j$, $\omega_k$ is presented as the sum of single-frequency matrices of positive combination frequencies $\Omega_1$, $\Omega_2$, $\Omega_3$, $\Omega_4$ :

$$\mathbf{\Phi}(\omega_i, \omega_j, \omega_k, t) =$$

$$= \mathbf{\Phi}(\Omega_1, \Omega_2, \Omega_3, \Omega_4, t) = \mathbf{\Phi}_1(\Omega_1, t) + \mathbf{\Phi}_2(\Omega_2, t) + \mathbf{\Phi}_3(\Omega_3, t) + \mathbf{\Phi}_4(\Omega_4, t), \quad (9)$$

where,

$$\mathbf{\Phi}_1(\Omega_1, t) = \frac{1}{4}\Big[\big(\cos(\Omega_1 t) + \sin(\Omega_1 t)\big)\mathbf{E} + \big(-\cos(\Omega_1 t) + \sin(\Omega_1 t)\big)\mathbf{I} + \quad (10)$$

$$+\big(\cos(\Omega_1 t) + \sin(\Omega_1 t)\big)\mathbf{J} + \big(-\cos(\Omega_1 t) + \sin(\Omega_1 t)\big),$$

$$\mathbf{\Phi}_2(\Omega_2, t) = \frac{1}{4}\Big[\big(\cos(\Omega_2 t) - \sin(\Omega_2 t)\big)\mathbf{E} + \big(\cos(\Omega_2 t) + \sin(\Omega_2 t)\big)\mathbf{I} +$$

$$+\big(-\cos(\Omega_2 t) + \sin(\Omega_2 t)\big)\mathbf{J} + \big(-\cos(\Omega_2 t) - \sin(\Omega_2 t)\big)\mathbf{K}\Big],$$

$$\mathbf{\Phi}_3(\Omega_3, t) = \frac{1}{4}\Big[\big(\cos(\Omega_3 t) - \sin(\Omega_3 t)\big)\mathbf{E} + \big(\cos(\Omega_3 t) + \sin(\Omega_3 t)\big)\mathbf{I} +$$

$$+\big(\cos(\Omega_3 t) - \sin(\Omega_3 t)\big)\mathbf{J} + \big(\cos(\Omega_3 t) + \sin(\Omega_3 t)\big)\mathbf{K}\Big],$$

$$\mathbf{\Phi}_4(\Omega_4, t) = \frac{1}{4}\Big[\big(\cos(\Omega_4 t) + \sin(\Omega_4 t)\big)\mathbf{E} + \big(-\cos(\Omega_4 t) + \sin(\Omega_4 t)\big)\mathbf{I} +$$

$$+\big(-\cos(\Omega_4 t) - \sin(\Omega_4 t)\big)\mathbf{J} + \big(\cos(\Omega_4 t) - \sin(\Omega_4 t)\big)\mathbf{K}\Big].$$

Since matrix $\mathbf{\Phi}(\omega_i, \omega_j, \omega_k, t)$ is orthogonal, matrix $\mathbf{\Phi}(\Omega_1, \Omega_2, \Omega_3, \Omega_4, t)$ will also be orthogonal.

The determinants of single-frequency matrices are calculated as

$$\left|\mathbf{\Phi}_n(\Omega_n, t)\right| = \cos^4(\Omega_n, t) + 2\cos^2(\Omega_n, t)\sin^2(\Omega_n, t) + \sin^4(\Omega_n, t) =$$

$$= \left(\cos^2(\Omega_n, t) + \sin^2(\Omega_n, t)\right)^2 = 1, \text{ where } n=1,2,3,4.$$

The matrices for the combination frequencies are also orthogonal, since

$\mathbf{\Phi}_n(\Omega_n, t)\mathbf{\Phi}_n^{\mathrm{T}}(\Omega_1, t) = \mathbf{\Phi}_n^{\mathrm{T}}(\Omega_1, t)\mathbf{\Phi}_n(\Omega_n, t) = \mathbf{E}$, where $n$=1,2,3,4.

Since, according to (9), we have obtained the equality of the fundamental matrix for the reference frequencies to the sum of four single-frequency matrices of combination frequencies, then the signal $\mathbf{y}(\Omega_1, \Omega_2, \Omega_3, \Omega_4, t)$ modulated by the information vector $\mathbf{x}(0)$ can also be obtained by multiplying this vector by each single-frequency matrix:

$$\mathbf{y}(\Omega_1, \Omega_2, \Omega_3, \Omega_4, t) = \quad (11)$$

$$= \mathbf{\Phi}(\omega_i, \omega_j, \omega_k, t)\mathbf{x}(0) = \left[\mathbf{\Phi}_1(\Omega_1, t) + \mathbf{\Phi}_2(\Omega_2, t) + \mathbf{\Phi}_3(\Omega_3, t) + \mathbf{\Phi}_4(\Omega_4, t)\right]\mathbf{x}(0) =$$

$$= \left[\mathbf{\Phi}_1(\Omega_1, t)\mathbf{x}(0) + \mathbf{\Phi}_2(\Omega_2, t)\mathbf{x}(0) + \mathbf{\Phi}_3(\Omega_3, t)\mathbf{x}(0) + \mathbf{\Phi}_4(\Omega_4, t)\mathbf{x}(0)\right] =$$

$$= \mathbf{y}_1(\Omega_1, t) + \mathbf{y}_2(\Omega_2, t) + \mathbf{y}_3(\Omega_3, t) + \mathbf{y}_4(\Omega_4, t).$$

As can be seen from (10) and (11), as a result of modulation we obtain the sum of phase-modulated subcarriers $\mathbf{y}_1(\Omega_1,t)$, $\mathbf{y}_2(\Omega_2,t)$, $\mathbf{y}_3(\Omega_3,t)$, $\mathbf{y}_4(\Omega_4,t)$. Examples of signals obtained using formula (11) are shown in [7].

When modulating by multiplying the information vector by the matrix (4) for reference frequencies, we obtain a signal vector, each element of which consists of the sum of different elements at different frequencies for the case $\omega = [6 \quad -2 \quad -1]^{\mathrm{T}}$, $\mathbf{\Omega} = [3 \quad 5 \quad 7 \quad 9]^{\mathrm{T}}$. For convenience of consideration, the frequencies of vectors ω and Ω are taken without multiplication by 2π.

Figure 1 shows a multi-frequency signal modulated by an information vector. $\mathbf{x}(0) = [1 \quad 1 \quad -1 \quad 1]$.

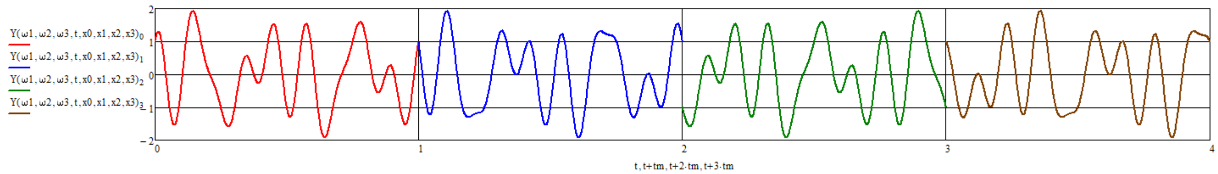


**Figure 1.** Multifrequency signal obtained by multiplying the initial vector x(0)=[11-11] by the 3-frequency fundamental matrix (4)

We obtain the same graphs by summing the elements of the vectors obtained using formula (11) by multiplying the information vector by single-frequency matrices of combination frequencies.

Figure 2 shows the graphs of the elements of the output modulated vectors (11) for the same combination frequencies considered above in the example, for the same information vector.

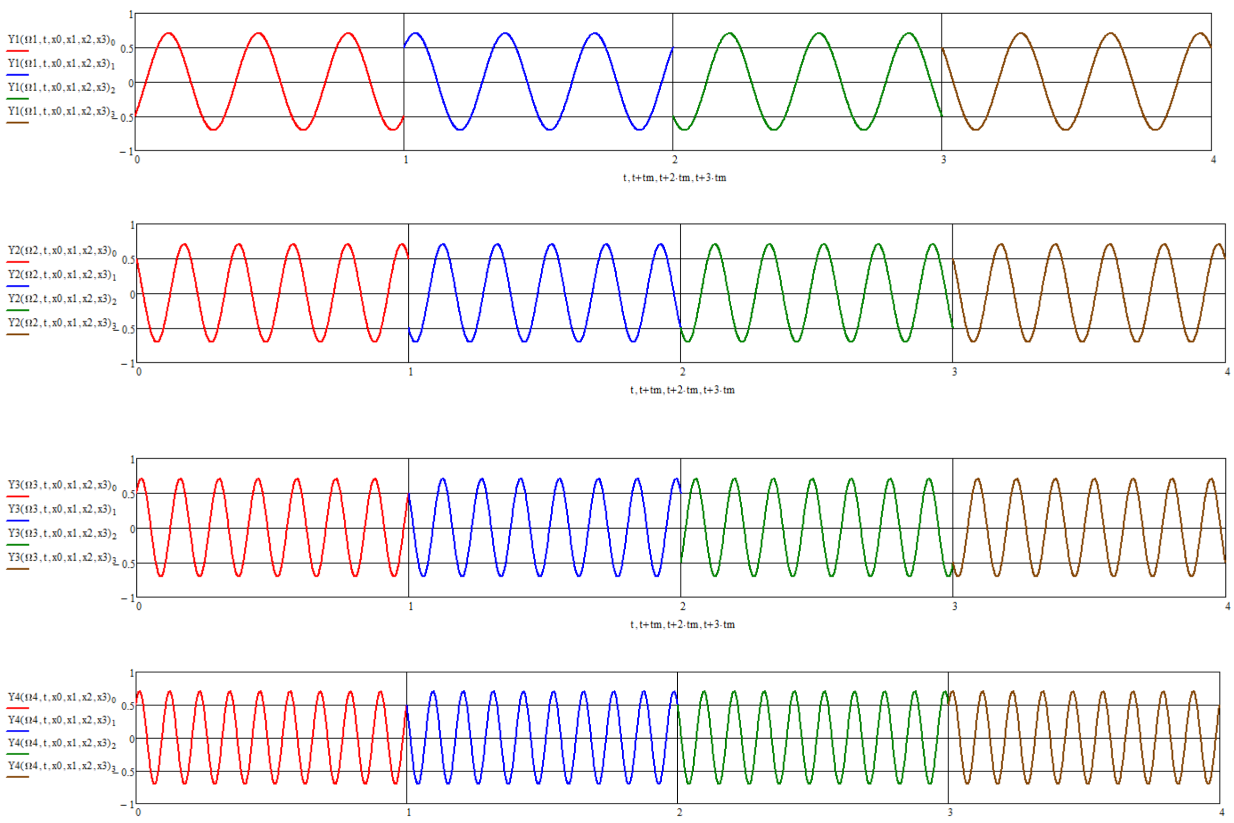


**Figure 2.** Phase-modulated signals obtained by multiplying the initial vector x(0)=[11-11] by single-frequency matrices of combination frequencies (10)

Using the obtained results, we will consider the method of calculating 3fQFT

### 3. Method of Obtaining a Three-Frequency Fourier Transform of a Vector of Pulses

Since the multi-frequency fundamental matrix (9) can be decomposed into four single-frequency matrices (10), to obtain 3fQFT we will use QFT with a transform kernel in the form of a single-frequency fundamental matrix for each combination frequency. The method for obtaining QFT is presented in [4]. The QFT and IQFT formulas for reference frequencies $\omega$ are:

$$\text{QFT:}\ \ \mathbf{g}(\omega)=\int_{-\infty}^{\infty}\mathbf{\Phi}^{\mathrm{T}}(\omega,t)\mathbf{q}(t)\,\mathrm{d}t\,, \tag{12}$$

$$\text{IQFT:}\ \ \mathbf{q}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{\Phi}(\omega,t)\mathbf{g}(\omega)\,\mathrm{d}\omega\,, \tag{13}$$

where $\mathbf{q}(t)=\left[x_0q_s(t)\quad x_1q_x(t)\quad x_2q_y(t)\quad x_3q_z(t)\right]^{\mathrm{T}}$ is the vector of analog pulses of a given shape, the sign and amplitude of which is determined by the information vector $\mathbf{x}(0)$, $\mathbf{g}(\omega)=\left[g_s(\omega)\quad g_x(\omega)\quad g_y(\omega)\quad g_z(\omega)\right]$ is the vector of spectra for the frequency $\omega$.

The indices of the vector elements correspond to the elements in the algebraic notation of the quaternion. The difference is that in [4] single-frequency matrices were considered with elements $\cos(\omega t)$ for the real part of the quaternion and $\sin(\omega t)$ for the imaginary part. A single-frequency quaternion can be viewed as a complex number with an imaginary unit $\hat{\mathbf{I}}=(\mathbf{I}+\mathbf{J}+\mathbf{K})/\sqrt{3}$ in matrix representation, only in 4D space. In this case, we consider three-frequency matrices (10) with elements $\pm\cos(\Omega_n t)\pm\sin(\Omega_n t)$ for combination frequencies, where $n$=1,2,3,4.

To simplify the calculations, we group the matrices (10) by cosines and sines and denote the resulting sums of the basis (5) matrices as $\hat{\mathbf{I}}_0=\mathbf{I}+\mathbf{J}+\mathbf{K}$, $\hat{\mathbf{I}}_1=-\mathbf{I}+\mathbf{J}-\mathbf{K}$, $\hat{\mathbf{I}}_2=\mathbf{I}-\mathbf{J}-\mathbf{K}$, $\hat{\mathbf{I}}_3=-\mathbf{I}-\mathbf{J}+\mathbf{K}$. These matrices have constant combination frequencies and we will use them to obtain the modulated output vector (11), so we will call them *modulating* and designate them with the index m:

$$\mathbf{\Phi}_{\mathrm{m},1}(\Omega_1,t)=\frac{1}{4}\left[\cos(\Omega_1 t)\left(\mathbf{E}+\hat{\mathbf{I}}_1\right)+\sin(\Omega_1 t)\left(\mathbf{E}+\hat{\mathbf{I}}_0\right)\right], \tag{14}$$

$$\mathbf{\Phi}_{\mathrm{m},2}(\Omega_2,t)=\frac{1}{4}\left[\cos(\Omega_2 t)\left(\mathbf{E}+\hat{\mathbf{I}}_2\right)+\sin(\Omega_2 t)\left(-\mathbf{E}-\hat{\mathbf{I}}_3\right)\right],$$

$$\mathbf{\Phi}_{\mathrm{m},3}(\Omega_3,t)=\frac{1}{4}\left[\cos(\Omega_3 t)\left(\mathbf{E}+\hat{\mathbf{I}}_0\right)+\sin(\Omega_3 t)\left(-\mathbf{E}-\hat{\mathbf{I}}_1\right)\right],$$

$$\mathbf{\Phi}_{\mathrm{m},4}(\Omega_4,t)=\frac{1}{4}\left[\cos(\Omega_4 t)\left(\mathbf{E}+\hat{\mathbf{I}}_3\right)+\sin(\Omega_4 t)\left(\mathbf{E}+\hat{\mathbf{I}}_2\right)\right].$$

Modulation of the quaternion carrier is performed, in accordance with (11), by multiplying the information vector $\mathbf{x}(0)$ by the modulating matrices (14):

$$\mathbf{y}_1(\Omega_1,t)=\mathbf{\Phi}_{\mathrm{m},1}(\Omega_1,t)\mathbf{q}(t)=\frac{1}{4}\left[\left(\mathbf{E}+\hat{\mathbf{I}}_1\right)\cos(\Omega_1 t)\mathbf{q}(t)+\left(\mathbf{E}+\hat{\mathbf{I}}_0\right)\sin(\Omega_1 t)\mathbf{q}(t)\right], \tag{15}$$

$$\mathbf{y}_2(\Omega_2,t)=\mathbf{\Phi}_{\mathrm{m},2}(\Omega_2,t)\mathbf{q}(t)=\frac{1}{4}\left[\left(\mathbf{E}+\hat{\mathbf{I}}_2\right)\cos(\Omega_2 t)\mathbf{q}(t)+\left(-\mathbf{E}-\hat{\mathbf{I}}_3\right)\sin(\Omega_2 t)\mathbf{q}(t)\right],$$

$$\mathbf{y}_3(\Omega_3,t)=\mathbf{\Phi}_{\mathrm{m},3}(\Omega_3,t)\mathbf{q}(t)=\frac{1}{4}\left[\left(\mathbf{E}+\hat{\mathbf{I}}_0\right)\cos(\Omega_3 t)\mathbf{q}(t)+\left(-\mathbf{E}-\hat{\mathbf{I}}_1\right)\sin(\Omega_3 t)\mathbf{q}(t)\right],$$

$$\mathbf{y}_4(\Omega_4,t)=\mathbf{\Phi}_{\mathrm{m},4}(\Omega_4,t)\mathbf{q}(t)=\frac{1}{4}\left[\left(\mathbf{E}+\hat{\mathbf{I}}_3\right)\cos(\Omega_4 t)\mathbf{q}(t)+\left(\mathbf{E}+\hat{\mathbf{I}}_2\right)\sin(\Omega_4 t)\mathbf{q}(t)\right],$$

where $\mathbf{y}_n(\Omega_n,t)=\left[y_{n,s}(\Omega_n,t)\quad y_{n,x}(\Omega_n,t)\quad y_{n,y}(\Omega_n,t)\quad y_{n,z}(\Omega_n,t)\right]^{\mathrm{T}}$ - vectors of modulated radio pulses on subcarriers $\Omega_n$, $n$=1,2,3,4.

In general, pulses $\mathbf{q}(t)$ can have different shapes [4,5]. Here we will consider rectangular pulses with the same amplitude and duration. For rectangular pulses, when multiplying the information vector $\mathbf{x}(0)$ by single-frequency matrices (14), the resulting vectors of the cosine and sine amplitudes are determined for the frequency $\Omega_1$ by the products $\mathbf{a}_{c,1}=\frac{1}{4}\left(\mathbf{E}+\hat{\mathbf{I}}_1\right)\mathbf{x}(0)$ and $\mathbf{a}_{s,1}=\frac{1}{4}\left(\mathbf{E}+\hat{\mathbf{I}}_0\right)\mathbf{x}(0)$, respectively. Also for frequency $\Omega_2$ : $\mathbf{a}_{c,2}=\frac{1}{4}\left(\mathbf{E}+\hat{\mathbf{I}}_2\right)\mathbf{x}(0)$ and $\mathbf{a}_{s,2}=\frac{1}{4}\left(-\mathbf{E}-\hat{\mathbf{I}}_3\right)\mathbf{x}(0)$, for frequency $\Omega_3$: $\mathbf{a}_{c,3}=\frac{1}{4}\left(\mathbf{E}+\hat{\mathbf{I}}_0\right)\mathbf{x}(0)$ and $\mathbf{a}_{s,3}=\frac{1}{4}\left(-\mathbf{E}-\hat{\mathbf{I}}_1\right)\mathbf{x}(0)$, for frequency $\Omega_4$: $\mathbf{a}_{c,4}=\frac{1}{4}\left(\mathbf{E}+\hat{\mathbf{I}}_3\right)\mathbf{x}(0)$ and $\mathbf{a}_{s,4}=\frac{1}{4}\left(\mathbf{E}+\hat{\mathbf{I}}_2\right)\mathbf{x}(0)$.

The corresponding phase vectors $\boldsymbol{\theta}_n$, where $n$=1,2,3,4, are calculated as the arctangent of the amplitude ratio. The found initial phases of the modulated signals for the corresponding combination frequency can be compared using the signals shown in Figure 2. For example, for the vector $\mathbf{x}(0)$ = [1 1 -1 1] and $\Omega_1$ we obtain the following values of amplitudes and phases:

$$\mathbf{a}_{c,1}=\left[a_{c,1,0}\quad a_{c,1,1}\quad a_{c,1,2}\quad a_{c,1,3}\right]^{\mathrm{T}}=\left[-0,5\quad 0,5\quad -0,5\quad 0,5\right]^{\mathrm{T}},$$

$$\mathbf{a}_{s,1}=\left[a_{s,1,0}\quad a_{s,1,1}\quad a_{s,1,2}\quad a_{s,1,3}\right]^{\mathrm{T}}=\left[0,5\quad 0,5\quad -0,5\quad -0,5\right]^{\mathrm{T}},$$

$$\boldsymbol{\theta}_1=\left[\theta_{1,0}\quad \theta_{1,1}\quad \theta_{1,2}\quad \theta_{1,3}\right]^{\mathrm{T}}=\left[135\quad 45\quad -135\quad -45\right]^{\mathrm{T}}.$$

To calculate 3fQFT for each combination frequency, we use formula (12). In this case, we will use transposed matrices corresponding to matrices (14) with index f:

$$\boldsymbol{\Phi}^{\mathrm{T}}_{\mathrm{f},1}(\Omega_{\mathrm{f},1},t)=\frac{1}{4}\left[\cos\left(\Omega_{\mathrm{f},1}t\right)\left(\mathbf{E}+\hat{\mathbf{I}}_1\right)^{\mathrm{T}}+\sin\left(\Omega_{\mathrm{f},1}t\right)\left(\mathbf{E}+\hat{\mathbf{I}}_0\right)^{\mathrm{T}}\right], \tag{16}$$

$$\boldsymbol{\Phi}^{\mathrm{T}}_{\mathrm{f},2}(\Omega_{\mathrm{f},2},t)=\frac{1}{4}\left[\cos\left(\Omega_{\mathrm{f},2}t\right)\left(\mathbf{E}+\hat{\mathbf{I}}_2\right)^{\mathrm{T}}+\sin\left(\Omega_{\mathrm{f},2}t\right)\left(-\mathbf{E}-\hat{\mathbf{I}}_3\right)^{\mathrm{T}}\right],$$

$$\boldsymbol{\Phi}^{\mathrm{T}}_{\mathrm{f},3}(\Omega_{\mathrm{f},3},t)=\frac{1}{4}\left[\cos\left(\Omega_{\mathrm{f},3}t\right)\left(\mathbf{E}+\hat{\mathbf{I}}_0\right)^{\mathrm{T}}+\sin\left(\Omega_{\mathrm{f},3}t\right)\left(-\mathbf{E}-\hat{\mathbf{I}}_1\right)^{\mathrm{T}}\right],$$

$$\boldsymbol{\Phi}^{\mathrm{T}}_{\mathrm{f},4}(\Omega_{\mathrm{f},4},t)=\frac{1}{4}\left[\cos\left(\Omega_{\mathrm{f},4}t\right)\left(\mathbf{E}+\hat{\mathbf{I}}_3\right)^{\mathrm{T}}+\sin\left(\Omega_{\mathrm{f},4}t\right)\left(\mathbf{E}+\hat{\mathbf{I}}_2\right)^{\mathrm{T}}\right].$$

As with the calculation of any FT, the frequency of the transformation kernel, in our case for all matrices, will take values from -∞ to +∞, and integration will be carried out over the duration of the signal.

Let us rewrite (12) for each combination frequency, $n$=1,2,3,4, according to the designations made in (14) and (16) for 3fQFT:

$$\text{3fQFT:}\quad \mathbf{g}_n(\Omega_n,\Omega_{\mathrm{f},n})\overset{QFT}{\leftarrow}\int_{-\infty}^{\infty}\boldsymbol{\Phi}^{\mathrm{T}}_{\mathrm{f},n}(\Omega_{\mathrm{f},n},t)\mathbf{y}_n(\Omega_n,t)\,\mathrm{d}t, \tag{17}$$

where modulated radio pulses $\mathbf{y}_n(\Omega_n,t)$ are calculated using formulas (15).

As a result of the calculations, we obtain spectral vectors for various combination frequencies (subcarriers) $\Omega_n$, $n$=1,2,3,4:

$$\mathbf{g}_n(\Omega_n,\Omega_{\mathrm{f},n})=\left[g_{n,s}(\Omega_n,\Omega_{\mathrm{f},n})\quad g_{n,x}(\Omega_n,\Omega_{\mathrm{f},n})\quad g_{n,y}(\Omega_n,\Omega_{\mathrm{f},n})\quad g_{n,z}(\Omega_n,\Omega_{\mathrm{f},n})\right]^{\mathrm{T}}.$$

Note that the frequencies $\Omega_{\mathrm{f},n}$ for all vectors typically range from -∞ to +∞. The spectrum obtained from a radio pulse $\mathbf{y}_n(\Omega_n,t)$ by integrating over time within the duration of the radio pulse for a given combination frequency $\Omega_n$ is a function of frequency $\Omega_{\mathrm{f},n}$.
We rewrite IQFT (13) according to the notations made (14), (16), in the form 3fIQFT:

$$\text{3fIQFT:}\ \ \mathbf{y}_n(\Omega_n,t) \underset{QFT^{-1}}{\leftarrow} \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{\Phi}_{\mathrm{f},n}(\Omega_{\mathrm{f},n},t)\mathbf{g}_n(\Omega_n,\Omega_{\mathrm{f},n})\,\mathrm{d}\Omega_{\mathrm{f},n}\,, \tag{18}$$

The signal found from the spectrum $\mathbf{g}_n(\Omega_n,\Omega_{\mathrm{f},n})$ for a given combination frequency $\Omega_n$ by integrating over the frequency $\Omega_{\mathrm{f},n}$ is a function of time.

Thus, it is shown that using the decomposition of a 3-frequency quaternion in a matrix representation, the fundamental matrix of which consists of products of cosines and sines of reference angular frequencies $\omega_i$, $\omega_j$, $\omega_k$ in different combinations, by the sum of cosines and sines of reference frequencies $\Omega_n$, $n$=1,2,3,4, it is possible to find the spectral characteristics of a 3-frequency quaternion using the known single-frequency quaternion Fourier transform (12), (13). The difference is that 3fQFT is calculated for radio pulses at combination frequencies. Moreover, for each combination frequency, its own fundamental matrix (16) is used as the transformation kernel.

Let us consider the question of finding the 3fQFT of radio pulses (15), obtained by multiplying the modulating matrix $\mathbf{\Phi}_{\mathrm{m},n}(\Omega_n,t)$, $n$=1,2,3,4, by the vectors of analog pulses $\mathbf{q}(t)$ with the amplitudes of the information vector $\mathbf{x}(0)$. Let us substitute the expressions for radio pulses (15) into 3fQFT (17) and write down the cosine and sine components separately:

$$\mathbf{g}_{c,1}(\Omega_1,\Omega_{\mathrm{f},1}) = \frac{1}{4}\int_{-\infty}^{\infty}\mathbf{\Phi}_{\mathrm{f},1}^{\mathrm{T}}(\Omega_{\mathrm{f},1},t)\left(\mathbf{E}+\hat{\mathbf{I}}_1\right)\cos\left(\Omega_1 t\right)\mathbf{q}(t)\,\mathrm{d}t\,, \tag{19}$$

$$\mathbf{g}_{s,1}(\Omega_1,\Omega_{\mathrm{f},1}) = \frac{1}{4}\int_{-\infty}^{\infty}\mathbf{\Phi}_{\mathrm{f},1}^{\mathrm{T}}(\Omega_{\mathrm{f},1},t)\left(\mathbf{E}+\hat{\mathbf{I}}_0\right)\sin\left(\Omega_1 t\right)\mathbf{q}(t)\,\mathrm{d}t\,,$$

$$\mathbf{g}_{c,2}(\Omega_2,\Omega_{\mathrm{f},2}) = \frac{1}{4}\int_{-\infty}^{\infty}\mathbf{\Phi}_{\mathrm{f},2}^{\mathrm{T}}(\Omega_{\mathrm{f},2},t)\left(\mathbf{E}+\hat{\mathbf{I}}_2\right)\cos\left(\Omega_2 t\right)\mathbf{q}(t)\,\mathrm{d}t\,,$$

$$\mathbf{g}_{s,2}(\Omega_2,\Omega_{\mathrm{f},2}) = \frac{1}{4}\int_{-\infty}^{\infty}\mathbf{\Phi}_{\mathrm{f},2}^{\mathrm{T}}(\Omega_{\mathrm{f},2},t)\left(-\mathbf{E}-\hat{\mathbf{I}}_3\right)\sin\left(\Omega_2 t\right)\mathbf{q}(t)\,\mathrm{d}t\,,$$

$$\mathbf{g}_{c,3}(\Omega_3,\Omega_{\mathrm{f},3}) = \frac{1}{4}\int_{-\infty}^{\infty}\mathbf{\Phi}_{\mathrm{f},3}^{\mathrm{T}}(\Omega_{\mathrm{f},3},t)\left(\mathbf{E}+\hat{\mathbf{I}}_0\right)\cos\left(\Omega_3 t\right)\mathbf{q}(t)\,\mathrm{d}t\,,$$

$$\mathbf{g}_{s,3}(\Omega_3,\Omega_{\mathrm{f},3}) = \frac{1}{4}\int_{-\infty}^{\infty}\mathbf{\Phi}_{\mathrm{f},3}^{\mathrm{T}}(\Omega_{\mathrm{f},3},t)\left(-\mathbf{E}-\hat{\mathbf{I}}_1\right)\sin\left(\Omega_3 t\right)\mathbf{q}(t)\,\mathrm{d}t\,,$$

$$\mathbf{g}_{c,4}(\Omega_4,\Omega_{\mathrm{f},4}) = \frac{1}{4}\int_{-\infty}^{\infty}\mathbf{\Phi}_{\mathrm{f},4}^{\mathrm{T}}(\Omega_{\mathrm{f},4},t)\left(\mathbf{E}+\hat{\mathbf{I}}_3\right)\cos\left(\Omega_4 t\right)\mathbf{q}(t)\,\mathrm{d}t\,,$$

$$\mathbf{g}_{s,4}(\Omega_4,\Omega_{\mathrm{f},4}) = \int_{-\infty}^{\infty}\mathbf{\Phi}_{\mathrm{f},4}^{\mathrm{T}}(\Omega_{\mathrm{f},4},t)\frac{1}{4}\left(\mathbf{E}+\hat{\mathbf{I}}_2\right)\sin\left(\Omega_4 t\right)\mathbf{q}(t)\,\mathrm{d}t\,.$$

Expressions (19) represent 3fQFT products of functions $\cos(\Omega_n t)$ and for combinational subcarriers $\Omega_n$ with $n$=1, 2, 3, 4, and the vector of analog pulses $\mathbf{q}(t)$. As is known, the spectrum of the product of cosines and sines on the elements of a vector corresponds to the spectrum of their convolution [10]. Therefore, to obtain spectra using formulas (19), it is sufficient to calculate the spectra of the analog elements of the vector $\mathbf{q}(t)$ and then perform the convolution of these spectra with the spectra of the sine and cosine. Since the sine and cosine spectra represent frequency delta pulses, the spectra of analog vector pulses $\mathbf{q}(t)$ are simply shifted along the frequency axis by the value of the subcarrier.

Thus, to calculate 3fQFT of known pulses, it is necessary to multiply a vector of four analog pulses $\mathbf{q}(t)$ by the corresponding basis matrices, then multiply the resulting vector by the transposed fundamental matrices for a given combination frequency and integrate over the pulse duration. The obtained spectra are shifted along the frequency axis by the value of the combination frequency. To understand what result will be obtained when calculating 3fQFT for a known signal, let us consider 3fQFT using formula (17), substituting expression (15) into it:

$$\mathbf{g}_n(\Omega_n,\Omega_{\mathrm{f},n}) \overset{QFT}{\leftarrow} \int_{-\infty}^{\infty} \mathbf{\Phi}^{\mathrm{T}}_{\mathrm{f},n}(\Omega_{\mathrm{f},n},t)\mathbf{\Phi}_{\mathrm{m},n}(\Omega_n,t)\mathbf{q}(t)\,\mathrm{d}t\,. \tag{20}$$

We multiply the matrices inside the integral (20) and as a result we obtain matrices, which we denote by indices f,m:

$$\begin{aligned}
\mathbf{\Phi}_{\mathrm{f,m},1}(\Omega_1,t) &= \mathbf{\Phi}^{\mathrm{T}}_{\mathrm{f},1}(\Omega_{\mathrm{f},1},t)\mathbf{\Phi}_{\mathrm{m},1}(\Omega_1,t) = \frac{1}{4}\Big[\cos[(\Omega_{\mathrm{f},1}-\Omega_1)t]\mathbf{E} - \sin[(\Omega_{\mathrm{f},1}-\Omega_1)t]\mathbf{K}\Big],\\
\mathbf{\Phi}_{\mathrm{f,m},2}(\Omega_2,t) &= \mathbf{\Phi}^{\mathrm{T}}_{\mathrm{f},2}(\Omega_{\mathrm{f},2},t)\mathbf{\Phi}_{\mathrm{m},2}(\Omega_2,t) = \frac{1}{4}\Big[\cos[(\Omega_{\mathrm{f},2}-\Omega_2)t]\mathbf{E} + \sin[(\Omega_{\mathrm{f},2}-\Omega_2)t]\mathbf{K}\Big],\\
\mathbf{\Phi}_{\mathrm{f,m},3}(\Omega_3,t) &= \mathbf{\Phi}^{\mathrm{T}}_{\mathrm{f},3}(\Omega_{\mathrm{f},3},t)\mathbf{\Phi}_{\mathrm{m},3}(\Omega_3,t) = \frac{1}{4}\Big[\cos[(\Omega_{\mathrm{f},3}-\Omega_3)t]\mathbf{E} - \sin[(\Omega_{\mathrm{f},3}-\Omega_3)t]\mathbf{K}\Big],\\
\mathbf{\Phi}_{\mathrm{f,m},4}(\Omega_4,t) &= \mathbf{\Phi}^{\mathrm{T}}_{\mathrm{f},4}(\Omega_{\mathrm{f},4},t)\mathbf{\Phi}_{\mathrm{m},4}(\Omega_4,t) = \frac{1}{4}\Big[\cos[(\Omega_{\mathrm{f},4}-\Omega_4)t]\mathbf{E} + \sin[(\Omega_{\mathrm{f},4}-\Omega_4)t]\mathbf{K}\Big].
\end{aligned} \tag{21}$$

According to the frequency shift property, modulating matrices $\mathbf{\Phi}_{\mathrm{m},n}(\Omega_n,t)$ simply shift the spectra of analog signals by combination frequencies $\Omega_n$. As can be seen from formulas (21), the frequency shift is to the right of zero, i.e., the spectrum exists only for positive frequencies. This is due to the fact that when calculating the spectra inside the 3fQFT integral we have a product of matrices that have the same structures for each frequency $\Omega_n$. Matrices (21) contain both cosine and sine components. The cosine components are located on the main diagonal, and the sine components are on the secondary diagonal. When matrix multiplication occurs, negative frequencies are compensated. When considering only cosine or sine carriers separately, as in (19), we get both positive and negative frequencies. As a result of adding the results, negative frequencies also cancel out, as do imaginary parts of the conjugate complex signals.

As is known, when transmitting information using the MIMO scheme, each pulse at the output of the communication channel is formed by summing all other input pulses. The calculation of the FT of the summed pulses for a single-frequency quaternion was considered in [4, 5].

The spectra of individual rectangular pulses with duration $T_i$ and symmetrical to zero of the time axes have the form $T_i$ sinc($T_i$ω/2) [10]. For a vector of rectangular pulses that are transmitted sequentially, there must be a shift in time positions $t_0 = T_\mathrm{i}/2$, $t_1 = 3T_\mathrm{i}/2$, $t_2 = 5T_\mathrm{i}/2$, $t_3 = 7T_\mathrm{i}/2$, accordingly. When adding up the vector pulses, we obtain a sequence of pulses following one another, and when adding up their spectra, we obtain the spectrum of a quaternion in the vector representation of a sequence of pulses that do not intersect in time. Since the phase relationships in the spectra change when the pulses shift over time, adding up the spectra produces a diffraction pattern [4].

Figure 3 shows the spectra of analog rectangular pulses when they are shifted in time for serial transmission. The spectrum of the 1st pulse in the vector is indicated by the red line, the 2nd by the blue line, the 3rd by the green line, and the 4th by the brown line. The spectrum of one pulse, symmetrical to the 0 time axis, is indicated by the turquoise line.

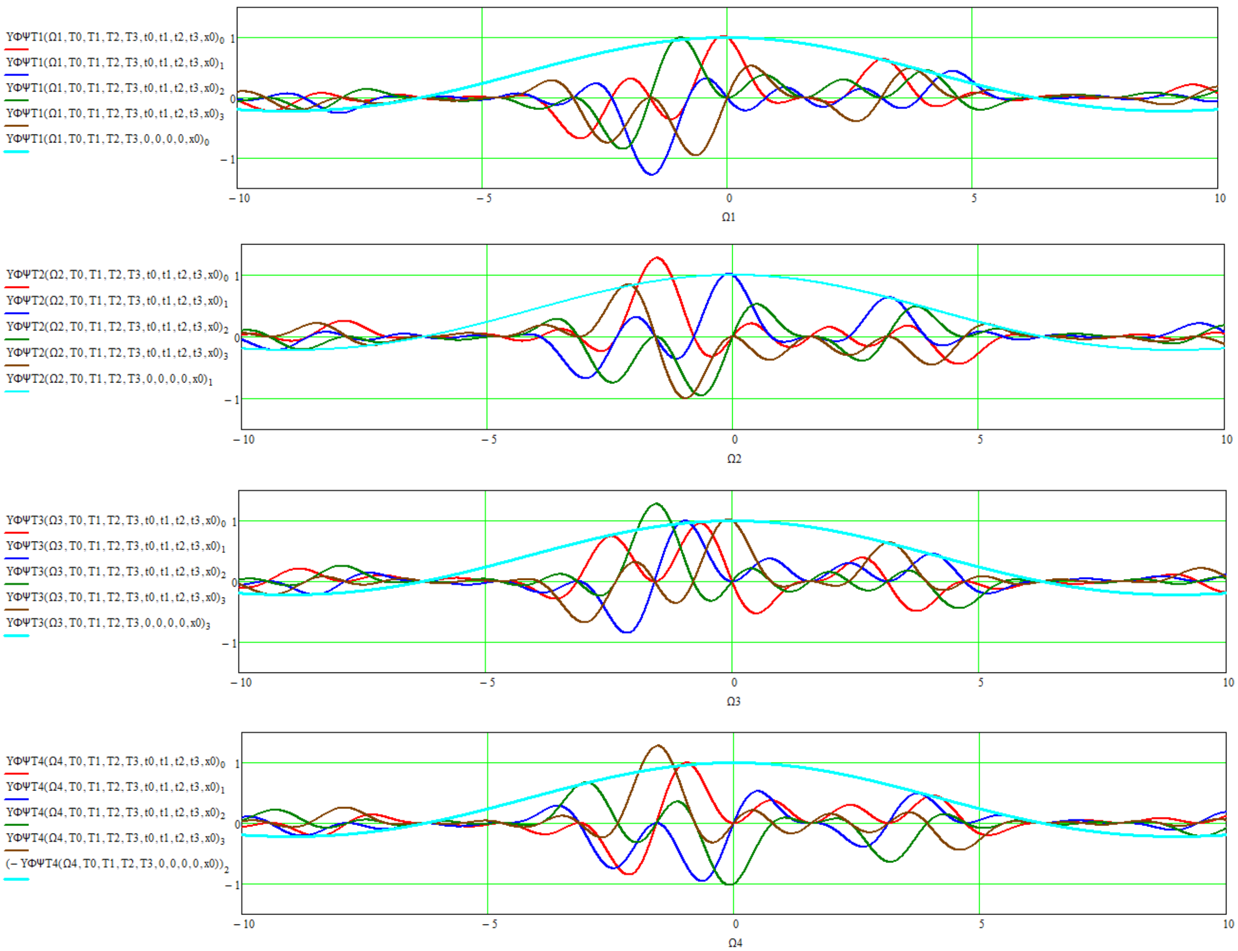


**Figure 3.** Spectra of the vector of analog rectangular pulses of the same length, during serial transmission, for different combination frequencies, with an amplitude corresponding to the value of the information vector **x**(0)=[11-11]

As can be seen from Figure 3, the spectrum of time-shifted pulses does not change in width for the first zeros of the spectrum of a single rectangular pulse, but only acquires oscillations in amplitude. Moreover, for each frequency the spectra of the same pulses differ, since the modulating matrices (14) differ. Since Parseval's equality for QFT is satisfied, when the shape of the spectrum changes during successive transmission of pulses, the energy of the pulse spectrum remains the same [4]. Since Parseval's equality for QFT is satisfied, when the shape of the spectrum changes during successive transmission of pulses, the energy of the spectrum vector of pulses remains the same [4].

To obtain 3fIQFT we use formula (18). The results of the transformation are shown in Figure 4.

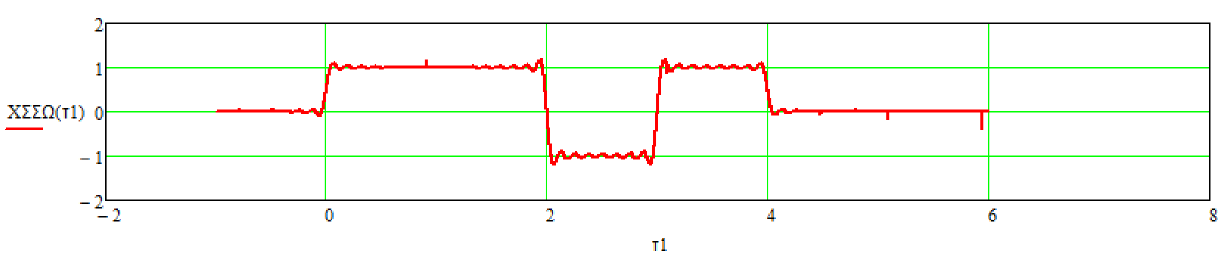


**Figure 4.** Inverse three-frequency quaternion Fourier transform of the spectrum of a rectangular pulse sequence.

As shown above, the frequencies of the three-frequency quaternion $\omega_i$, $\omega_j$, $\omega_k$ are located in space on three orthogonal coordinate axes *i, j, k*. Therefore, just as the signals in Figure 2 were depicted in space 3D, the spectra of these signals can also be depicted [7]. To do this, using the transformation matrix (7), we calculate the change in the reference frequencies $\omega_i$, $\omega_j$, $\omega_k$ when changing the combination frequencies $\Omega_n$, $n$=1,2,3,4. For example, when changing the combination frequency $\Omega_1$ from -∞ to +∞, we obtain the values of changes in the reference frequencies for the case $\boldsymbol{\omega}=[6 \quad -2 \quad -1]^{\mathrm{T}}$, $\boldsymbol{\Omega}=[3 \quad 5 \quad 7 \quad 9]^{\mathrm{T}}$:

$$\frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\end{bmatrix}\begin{bmatrix}\Omega_1\\ 5\\ 7\\ 9\end{bmatrix}=\begin{bmatrix}\Omega_1+5+7+9\\ \Omega_1+5-7-9\\ \Omega_1-5+7-1\end{bmatrix}=\begin{bmatrix}\Omega_1+21\\ \Omega_1-11\\ \Omega_1+1\end{bmatrix}=\begin{bmatrix}\omega_i(\Omega_1)\\ \omega_j(\Omega_1)\\ \omega_k(\Omega_1)\end{bmatrix}.$$

Figure 5 shows the dependences of the values of reference frequencies on the change in combination frequencies $\Omega_n$, $n$ = 1,2,3,4 . The changes $\omega_i$ are shown by the red line, the $\omega_j$ - blue line, and the $\omega_k$ - green line.

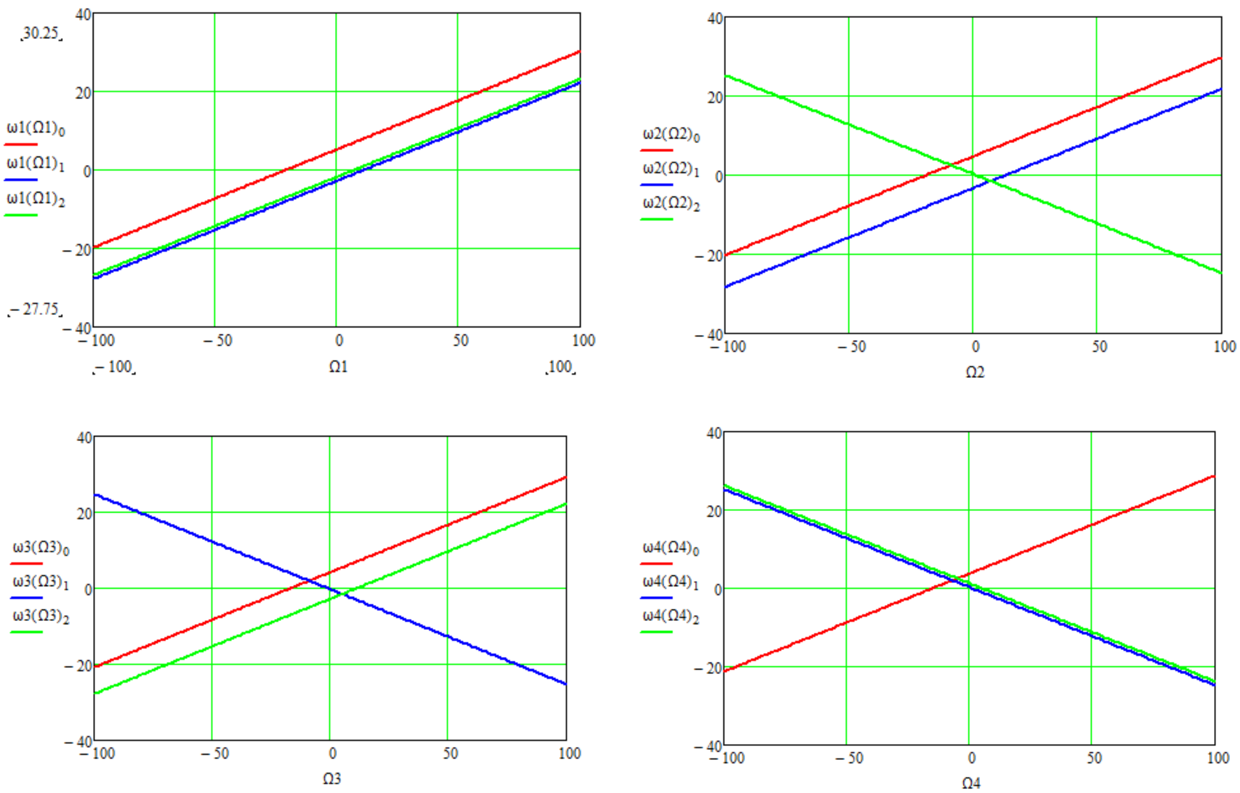


**Figure 5.** Graphs of the dependence of reference frequencies $\omega_i$, $\omega_j$, $\omega_k$ when changing combination frequencies $\Omega_n$, $n$=1,2,3,4

Figure 6 shows the spectra depicted in Figure 3 for each frequency in the 3D frequency space of the reference frequencies. Since the quaternion, considering the scalar part, has a dimension of 4D, the amplitude of the spectra is depicted using the value of the point on the graph. Positive values are highlighted in red and negative values are highlighted in blue. In this case, each graph shows the spectra of all pulses. However, it is possible to show each pulse separately for each frequency.

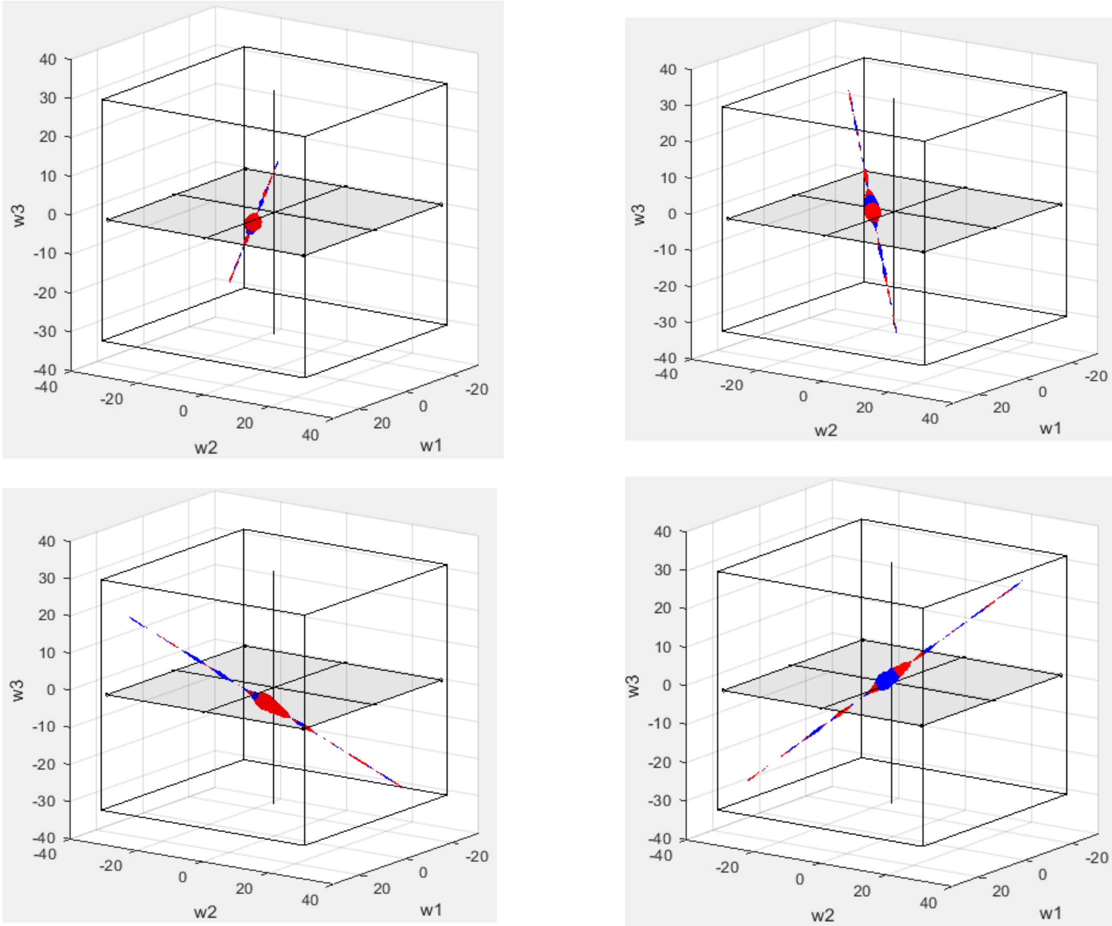


**Figure 6.** Spectra of rectangular pulses transmitted sequentially in the space of reference frequency values

As can be seen from Figure 6, the spectra have different orientations in space and, consequently, the volume of frequency space and the efficiency of its use increase. In addition, when calculating the values of the reference frequencies for the zero values of the combination frequencies, it is shown that these values do not intersect:

$$\begin{bmatrix}\omega_i(\Omega_1)\\ \omega_j(\Omega_1)\\ \omega_k(\Omega_1)\end{bmatrix}_{\Omega_1=0}=\begin{bmatrix}5.25\\ -2.75\\ -1.75\end{bmatrix},\ \begin{bmatrix}\omega_i(\Omega_2)\\ \omega_j(\Omega_2)\\ \omega_k(\Omega_2)\end{bmatrix}_{\Omega_2=0}=\begin{bmatrix}4.75\\ -3.25\\ 0.25\end{bmatrix},\ \begin{bmatrix}\omega_i(\Omega_3)\\ \omega_j(\Omega_3)\\ \omega_k(\Omega_3)\end{bmatrix}_{\Omega_3=0}=\begin{bmatrix}4.25\\ -0.25\\ -2.75\end{bmatrix},$$

$$\begin{bmatrix}\omega_i(\Omega_4)\\ \omega_j(\Omega_4)\\ \omega_k(\Omega_4)\end{bmatrix}_{\Omega_4=0}=\begin{bmatrix}3.75\\ 0.25\\ 1.25\end{bmatrix}.$$

It follows that the maximum values of the spectra also do not influence each other.

## 4. Conclusion

Existing mathematical methods for calculating the spectra of quaternion signals are based on calculating volume integrals in 4D space. In this case, as a rule, a single-frequency quaternion is used. Calculating a volume integral in 4D space causes certain difficulties and does not allow obtaining formulas for the spectra. The proposed method for calculating the spectra of 3-frequency quaternion signals is based on the representation of the Fourier transform kernel of a quaternion in matrix form. In this case, the obtained products of cosines and sines of reference frequencies in different combinations are expanded into the sum of cosines and sines of the total reference frequencies, which form combination frequencies. Using this decomposition, the 3-frequency quaternion Fourier transform is calculated as the sum of the single-frequency Fourier transforms of the combination frequencies.

Using the connection of combination frequencies with reference frequencies, the possibility of representing the spectra of quaternion signals in 3D space is shown. This representation allows us to conclude that quaternion signals use the 3D frequency space more efficiently.

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