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# Tensor Lorentz Force Representation Yields a New Electromagnetic Canonical Energy-Momentum Tensor

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**Abstract**
*As is known, the Maxwell stress-energy tensor typically used in electromagnetism is not a canonical tensor in the sense of Noether, since its four-divergence is not zero in the presence of sources, in other words, outside of free space. Indeed, the result provided by the calculation of the four-divergence of this tensor is the opposite of the four-vector generalizing the Lorentz force density. The idea is to associate with it a tensor constructed from potentials and sources, such that its four-divergence is the opposite of that of the Maxwell tensor. The distribution of sources will also be analyzed in light of fluid mechanics, allowing us to account for its influence in terms of generated pressure which maybe could overcome the fluid pressure leading to negative pressure and possibly negative energy in the case of hyper-relativistic fluids.*

**Keywords:** Theoretical Physics, Electromagnetism, Cosmology, General Relativity, Noether, Noether Symmetries, Energy-Momentum Tensor

## 1. Introduction

In the context of Noether's theorem, an energy-momentum tensor is canonical if the four-divergence is zero. This corresponds to the conservation of energy and momentum. Unfortunately, the tensor commonly used for electromagnetism satisfies this condition only in the absence of charges and currents.

If, within the framework of general relativity, we wish to have a geometrically correct description of spacetime on scales of galaxies, star systems, etc., we cannot neglect the influence of the charged particles present and their interactions with electromagnetic fields. Since the fields derive from scalar and vector potentials, this is equivalent to expressing these interactions in charge/potential form. These phenomena must therefore be taken into account in the energy-momentum tensor that appears in Einstein's equations.

To compensate for this deficiency in the Maxwell tensor, we will construct, "by hand", a complementary symmetric tensor in such a way that the overall four-divergence is zero, whether in the presence or absence of sources. If sources are present, they will be modeled by a fluid of particles. It will be seen that symmetry comes at a cost: the potentials must satisfy the Lorenz gauge.

For the sake of simplicity in writing, the development is first carried out within the framework of special relativity, in Minkowski spacetime, and will then be extended to general relativity.

## 2. Particle Distribution

We consider a continuous distribution of particles where, for the sake of simplicity, all the particles are the same. This distribution can be likened to an incompressible ideal relativistic fluid. As a consequence, its rest mass density $\rho_m$ is constant and the speed divergence is zero $\partial_\mu v^\mu = 0$.

The quantities m and q being constants yields the ratio $\frac{\rho_e}{\rho_m} = \frac{q}{m}$.
The expression for the canonical relativistic energy-momentum tensor can be found in[1,2,3,4] and is

$$T_{fl}^{\mu\nu} = (\mathcal{E} + P)\, u^{\mu} u^{\nu} - \eta^{\mu\nu} P \tag{1}$$

with the symbols defined previously [1-4].

The conservation laws of energy and momentum imply that

$$\partial_{\mu} T_{fl}^{\mu\nu} = 0$$

The evidence that this relationship is verified is available in [1-4].

## 3. Initial Tensor

In this section, we will attempt to construct an initial tensor whose four-divergence provides the generalized Lorentz force density $\left(\frac{\boldsymbol{E.J}}{c}, \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}\right)$. It is observed that it is possible to obtain it by taking the four-divergence of a tensor $\Lambda^{\mu\nu}\,(\phi_{\alpha})$ whose components are defined as

$$\Lambda^{00} = -\boldsymbol{A.J}$$
$$\Lambda^{i0} = -j^i \phi^0$$
$$\Lambda^{0i} = -j^0 \phi^i$$
$$\Lambda^{ii} = -\rho_e \Phi + \boldsymbol{A}.J - j^i \phi^i$$
$$\Lambda^{ij} = -j^i \phi^j$$

or, more simply, as

$$\Lambda^{\mu\nu}\,(\phi_{\alpha}) = -j^{\mu}\phi^{\nu} + \eta^{\mu\nu}\phi_{\alpha} j^{\alpha} \tag{2}$$

This tensor will be made symmetric later.

## 4. Trace

In Minkowski spacetime, the trace is readily calculated, which leads to

$$\Lambda_{\mu}^{\mu} = 3\phi_{\alpha} j^{\alpha}$$

where $\phi_{\alpha} j^{\alpha}$, being the dot product of two 4-vectors, is a relativistic invariant.
The result of calculating the trace immediately brings to mind the term of pressure obtained in the case of a fluid. We will revisit this later in the context of this paper.

## 5. Angular Momentum Conservation

If the canonical tensor $\Lambda^{\mu\nu}$ is not symmetric, the angular momentum tensor is written as

$$\mathcal{M}^{\alpha\mu\nu} = x^{\mu}\Lambda^{\alpha\nu} - x^{\nu}\Lambda^{\alpha\mu} + S^{\alpha\mu\nu}$$

where $S^{\alpha\mu\nu}$ is an intrinsic spin tensor. By construction $S^{\alpha\mu\nu}$ is anti-symmetric in $(\mu, \nu)$.
The angular momentum conservation implies $\partial_{\alpha} M^{\alpha\mu\nu} = 0$ and thus, taking into account that $\partial_{\alpha}\Lambda^{\alpha\mu} = \partial_{\alpha}\Lambda^{\alpha\nu} = 0$, the relation $\Lambda^{\mu\nu} - \Lambda^{\nu\mu} = \partial\alpha S\alpha\nu\mu$ has to be satisfied.

The anti-symmetric part of the tensor to be made symmetric must therefore be expressible as the four-divergence of a rank-3 tensor if one wishes to make it symmetric.

If we examine the tensor $\Lambda^{\mu\nu}$, the only part to make symmetric is $-j^{\mu}\phi^{\nu}$.

The calculation of the anti-symmetric part yields

$$\Lambda^{\mu\nu}-\Lambda^{\nu\mu}=\frac{1}{\mu_0}\left[\partial_\alpha\left(\phi^\mu F^{\alpha\nu}-\phi^\nu F^{\alpha\mu}\right)+\partial_\alpha\left(\phi^\mu\partial^\nu\phi^\alpha-\phi^\nu\partial^\mu\phi^\alpha\right)+\phi^\nu\partial_\alpha\partial^\mu\phi^\alpha-\phi^\mu\partial_\alpha\partial^\nu\phi^\alpha\right]$$

where $F_{\mu\alpha}=\partial_\mu\phi_\alpha-\partial_\alpha\phi_\mu$where $F^{\mu\alpha}=\partial^\mu\phi^\alpha-\partial^\alpha\phi^\mu$.

Examination of this relationship shows that the only way to express it solely in the form of a four-divergence involves $\partial_\alpha\phi^\alpha=0$, the Lorenz gauge.

Under this condition, we finally obtain

$$S^{\alpha\nu\mu}=\frac{1}{\mu_0}\left(\phi^\mu\partial^\alpha\phi^\nu-\phi^\nu\partial^\alpha\phi^\mu\right)$$

We will now apply the Belinfante method [5].

Let us define $\theta^{\alpha\mu\nu}=\frac{1}{2}\left(S^{\alpha\nu\mu}+S^{\nu\alpha\mu}-S^{\mu\nu\alpha}\right)$ which must be anti-symmetric in the indices $(\alpha,\mu)$ to preserve $\partial_\mu\Lambda^{\mu\nu}=0$; its computation yields

$$\theta^{\alpha\mu\nu}=\frac{1}{2\mu_0}\left(\phi^\mu\partial^\alpha\phi^\nu+\phi^\mu\partial^\nu\phi^\alpha+\phi^\nu\partial^\mu\phi^\alpha-\phi^\nu\partial^\alpha\phi^\mu-\phi^\alpha\partial^\mu\phi^\nu-\phi^\alpha\partial^\nu\phi^\mu\right)$$

The final symmetric canonical stress-energy-momentum tensor $\Theta^{\mu\nu}\left(\phi_\alpha,\partial_\mu\phi_\alpha\right)$ is obtained by subtracting $\partial_\alpha\theta^{\alpha\mu\nu}$ from “equation (2)” and adding the Maxwell energy-momentum tensor $\frac{1}{\mu_0}F^\mu_\alpha F^{\alpha\nu}+\frac{\eta^{\mu\nu}}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta}$ yielding

$$\begin{aligned}\Theta^{\mu\nu}\left(\phi_\alpha,\partial_\mu\phi_\alpha\right)&=\frac{1}{\mu_0}F^\mu_\alpha F^{\alpha\nu}+\frac{\eta^{\mu\nu}}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta}-\frac{1}{2}\phi^\mu j^\nu-\frac{1}{2}\phi^\nu j^\mu+\eta^{\mu\nu}j_\alpha\phi^\alpha\\&+\frac{1}{2\mu_0}\partial_\alpha\left(\phi^\alpha\partial^\mu\phi^\nu+\phi^\alpha\partial^\nu\phi^\mu-\phi^\mu\partial^\nu\phi^\alpha-\phi^\nu\partial^\mu\phi^\alpha\right)\end{aligned}$$

Even thoughtheterm $\partial_\alpha\left(\phi^\alpha\partial^\mu\phi^\nu+\phi^\alpha\partial^\nu\phi^\mu-\phi^\mu\partial^\nu\phi^\alpha-\phi^\nu\partial^\mu\phi^\alpha\right)$ is afour-divergence,itcannot be eliminatedinanyway.Ifitwere, $\partial_\mu\Theta^{\mu\nu}=0$ wouldnolongerbesatisfied.

### 6. Hamiltonian Electromagnetic Density

For the electromagnetic part, the total energy is provided by the Hamiltonian density. In the case of fields, it is given by the relation $\mathcal{H}=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_\alpha)}\partial^0\phi_\alpha-\mathcal{L}$. Its application to $-\frac{1}{4\mu_0}F_{\mu\alpha}F^{\mu\alpha}-\phi_\alpha j^\alpha$ results in

$$\begin{aligned}\mathcal{H}&=-\frac{1}{\mu_0}F^{0\alpha}\partial^0\phi_\alpha+\frac{1}{4\mu_0}F_{\mu\alpha}F^{\mu\alpha}+\phi_\alpha j^\alpha\\&=\epsilon_0E^2+\epsilon_0\boldsymbol{E}.\nabla\Phi-\frac{1}{2}\left(\epsilon_0E^2-\frac{B^2}{\mu_0}\right)+\rho_e\Phi-\boldsymbol{A}.\boldsymbol{J}\\&=\frac{1}{2}\left(\epsilon_0E^2+\frac{B^2}{\mu_0}\right)-\epsilon_0\Phi\nabla.\boldsymbol{E}+\rho_e\Phi-\boldsymbol{A}.\boldsymbol{J}+\nabla.\left(\epsilon_0\Phi\boldsymbol{E}\right)\\&=\frac{1}{2}\left(\epsilon_0E^2+\frac{B^2}{\mu_0}\right)-\boldsymbol{A}.\boldsymbol{J}+\nabla.\left(\epsilon_0\Phi\boldsymbol{E}\right)\end{aligned}$$

Upon integration over all space, the divergence presents in the above expression gives no contribution, being transformed into a surface integral at infinity where all fields and potentials are identically zero. This term vanishes and we obtain finally

$$\mathcal{H} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) - \boldsymbol{A}.\boldsymbol{J}$$

In our canonical tensor, the Hamiltonian density is provided by the term $\Theta^{00}$. Calculating this element leads to the result

$$\Theta^{00} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) - \boldsymbol{A}.\boldsymbol{J} + \frac{1}{\mu_0}\partial_\alpha\left(\phi^\alpha \partial^0 \phi^0 - \phi^0 \partial^0 \phi^\alpha\right)$$

which corresponds well to the expected energy density as the four-divergence gives no contribution upon integration over all space and therefore vanishes.

## 7. The Complete SEM Tensor

Until now, our tensor only refers to the electromagnetic fields and their interaction with matter. To be complete, we must add the tensor related to matter defined by "equation (1)". This provides the overall tensor

$$\begin{aligned} T^{\mu\nu}\left(\phi_\alpha, \partial_\mu \phi_\alpha, u_\alpha\right) = \frac{1}{\mu_0} F^\mu_\alpha F^{\alpha\nu} + \frac{\eta^{\mu\nu}}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta} + (\mathcal{E} + P)\, u^\mu u^\nu \\ -\frac{1}{2}\phi^\mu j^\nu - \frac{1}{2}\phi^\nu j^\mu + \eta^{\mu\nu}\left(j_\alpha \phi^\alpha - P\right) \\ + \frac{1}{2\mu_0}\partial_\alpha\left(\phi^\alpha \partial^\mu \phi^\nu + \phi^\alpha \partial^\nu \phi^\mu - \phi^\mu \partial^\nu \phi^\alpha - \phi^\nu \partial^\mu \phi^\alpha\right) \end{aligned} \quad (3)$$

Even though the term $\partial_\alpha\left(\phi^\alpha \partial^\mu \phi^\nu + \phi^\alpha \partial^\nu \phi^\mu - \phi^\mu \partial^\nu \phi^\alpha - \phi^\nu \partial^\mu \phi^\alpha\right)$ is a four-divergence, it cannot be eliminated in any way. If it were, $\partial_\mu T^{\mu\nu} = 0$ would no longer be satisfied.

This is result is similar to the tensor developed in [6].

## 8. Hamiltonian Density

In our canonical tensor, the Hamiltonian density is provided by the term $T^{00}$. Calculating this element leads to the following expression:

$$T^{oo} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) - \boldsymbol{A}.\boldsymbol{J} + \frac{\mathcal{E} + P\frac{V^2}{c^2}}{1 - \frac{V^2}{c^2}}$$

which corresponds well to the sum of the electromagnetic energy and the fluid energy densities.

To find the nonrelativistic limit ($v << c$), it is convenient to express the total energy density E as the sum of the mass energy $\rho_m c^2$ and the internal energy $\rho_m \varpi$ densities.

In doing so, it brings us to

$$T^{oo} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) - \boldsymbol{A}.\boldsymbol{J} + \rho_m c^2 + \rho_m \varpi + \frac{1}{2}\rho_m v^2$$

where the contribution of $P\frac{V^2}{c^2}$ is not taken into account as negligible with respect to the other terms.

In a comoving frame, as $v = 0$, $A.\ J = \rho_e A.v = 0$ and the Hamiltonian density reduces to $T^{00} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) + \mathcal{E}$.

## 9. $T^\mu_{\ \mu}$ in Special Relativity

In Minkowski spacetime, $T^\mu_{\ \mu}$ is readily calculated, which leads to

$$T^\mu_{\ \mu} = \mathcal{E} - 3\left(P - \phi_\alpha j^\alpha\right)$$

where $\phi_\alpha j^\alpha$, which is the dot product of two 4-vectors, is a relativistic invariant.

In a comoving frame, as $v = 0$, $\phi_\alpha j^\alpha$ reduces to $\rho_e \Phi$ and $T^\mu_{\ \mu}$ reduces to $\varepsilon - 3\ (P - \rho_e \Phi)$.

In astrophysical scenarios, relativistic fluids are often encountered in extreme environments, such as close to massive objects like black holes or in high-energy events like supernovae. In these situations, the speeds of particles or matter involved are comparable to the speed of light.

For a relativistic perfect fluid, one can demonstrate [6] that the trace of its energy-momentum tensor equals $\gamma^{-1}\rho_m c^2$.

As for ultra-relativistic fluids $\gamma^{-1} \approx 0$ and hence $\varepsilon \approx 3P$. It means that $T^\mu_{\ \mu}$ will then depends almost only on the electromagnetism through $3\phi_\alpha j^\alpha$ or $3\rho_e \Phi$ in a co-moving rest frame.

For baryonic matter, $T^\mu_{\ \mu}$ must be positive or null which implies $\varepsilon \geq 3\ (P - \phi_\alpha j^\alpha)$. Violation of this condition would indicate that we are in the presence of exotic matter.

As it's always possible to find a region of space small enough to use the usual Minkowski spacetime coordinates, it would be interesting to check whether $3\phi_\alpha j^\alpha$ does not become (transiently) negative enough for $T^\mu_{\ \mu}$ to become negative too, especially in regions where ultrarelativistic fluids are present.

## 10. General Relativity

In the presence of charges, the Maxwell stress-energy momentum tensor is not suitable for use, as is the case for Einstein's equation, because its 4-divergence is not null. On the other hand, the tensor that we developed in (3) does not suffer from this defect and can be introduced directly, as is, in Einstein's equation.

Its covariant form, adapted to general relativity, is given by

$$\begin{aligned} T_{\mu\nu}\left(\phi_\alpha, \partial_\mu\phi_\alpha, u_\alpha\right) = &\frac{1}{\mu_0}F_{\mu\alpha}F^\alpha_{\ \nu} + \frac{g_{\mu\nu}}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta} + (\mathcal{E} + P)\,u_\mu u_\nu \\ &-\frac{1}{2}\phi_\mu j_\nu - \frac{1}{2}\phi_\nu j_\mu + g_{\mu\nu}\left(\phi_\alpha j^\alpha - P\right) \\ &-\frac{1}{2\mu_0}\nabla_\alpha\left(\phi_\mu\nabla_\nu\phi^\alpha + \phi_\nu\nabla_\mu\phi^\alpha - \phi^\alpha\nabla_\mu\phi_\nu - \phi^\alpha\nabla_\nu\phi_\mu\right) \end{aligned} \quad (4)$$

that canbesplitintotheelectromagneticpart

$$\begin{aligned} T_{EM\,\mu\nu}\left(\phi_\alpha, \partial_\mu\phi_\alpha\right) = &\frac{1}{\mu_0}F_{\mu\alpha}F^\alpha_{\ \nu} + \frac{g_{\mu\nu}}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}\phi_\mu j_\nu - \frac{1}{2}\phi_\nu j_\mu + g_{\mu\nu}\phi_\alpha j^\alpha \\ &-\frac{1}{2\mu_0}\nabla_\alpha\left(\phi_\mu\nabla_\nu\phi^\alpha + \phi_\nu\nabla_\mu\phi^\alpha - \phi^\alpha\nabla_\mu\phi_\nu - \phi^\alpha\nabla_\nu\phi_\mu\right) \end{aligned} \quad (5)$$

and the fluid part $T_{fl\,\mu\nu}\left(u_\alpha\right) = (\mathcal{E} + P)\,u_\mu u_\nu - g_{\mu\nu}P$. This is result is similar to the tensor developed in [6].

## 11. Conclusions

The motivation of this article was to render the four-divergence of the Maxwell energy-momentum tensor zero in the presence of charges. This is the result we present here, achieved by introducing a tensor physically connected to fluid mechanics, where the sources are necessarily a consequence of the presence of massive particles.

The main idea was to find out a tensor which its four-divergence yields the Lorentz density four-vector and augmented the Maxwell tensor in order to make it canonical, meaning that its four-divergence is zero in the presence of charged or uncharged massive particles.

The final result for electromagnetism is given by "equation 5".

By treating these particles as an incompressible fluid, we observe that these sources have the effect of modifying the fluid pressure, as it could be calculated, for instance, from the ideal gas law.

We also open some paths to detect exotic matter in vicinity of black holes and supernovae. Clearly, this modeling is only a basic example, and further modeling can be envisaged by adding additional tensors, provided that they agree with Einstein's equation of general relativity.

By treating these particles as incompressible fluids, we observe that these sources modify the fluid pressure, as can be calculated, for instance, from the ideal gas law.

We hope that this article will contribute to better modeling ionized gas behavior within the interstellar medium and, linked to metrics such as the Schwarzschild or Friedmann–Lematre–Robertson–Walker metrics, improve cosmology understanding.

The price to pay for achieving tensor symmetry is the adherence to the Lorenz gauge. Therefore, it is tempting to consider the Lorenz gauge as a fifth equation of electromagnetism that would complement the four Maxwell's equations.

**Notation**

The Greek indices take the values 0, 1, 2, 3, while the Latin indices range from 1 to 3.
The main symbols used in this paper are summarized in the following table.

| | |
|---|---|
| $\eta^{\mu\nu} = \eta_{\mu\nu} = diag\,(1, -1, -1, -1)$ | Minkowski metric |
| $g_{\mu\nu}$ | General relativity spacetime metric |
| $\boldsymbol{E}$ | Electric field |
| $\boldsymbol{B}$ | Magnetic induction |
| $q$ | Particle charge |
| $m$ | Particle mass |
| $P$ | Fluid pressure |
| $n$ | Particle density (number of particles per $m^3$) |
| $\boldsymbol{x}$ | Position in 3D |
| $x^\mu = (ct, \boldsymbol{x})$ | Position 4-vector |
| $\nabla = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$ | 3D Nabla operator |
| $\partial_\mu = \left(\frac{\partial}{\partial x^0}, \nabla\right)$ | 4-divergence derivative operator |
| $\nabla_\mu$ | 4-divergence covariant derivative operator |
| $\boldsymbol{v}$ | Speed in 3D |
| $v^\mu = (c, \boldsymbol{v})$ | Speed 4-vector |
| $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ | Lorentz contraction factor |
| $u^\mu = \gamma\left(1, \frac{\boldsymbol{v}}{c}\right)$ | Normalized speed 4-vector |
| $\rho_m = mn$ | Particle rest mass density |
| $\rho_e = qn$ | Particle rest charge density |
| $\mathcal{E}$ | Particle rest total energy density |
| $\boldsymbol{J} = \rho_e \boldsymbol{v}$ | Current density |
| $\Phi$ | Scalar potential |
| $\boldsymbol{A}$ | Vector potential |

| | |
|---|---|
| $\phi^{\mu} = \left(\frac{\Phi}{c}, \boldsymbol{A}\right)$ | Potential 4-vector |
| $F^{\mu\nu} = \partial^{\mu}\phi^{\nu} - \partial^{\nu}\phi^{\mu}$ | Contravariant Faraday tensor |
| $j^{\mu} = (\rho_e c, \rho_e \boldsymbol{v}) = \frac{1}{\mu_0}\partial_{\alpha}F^{\alpha\mu}$ | Current density 4-vector |
| $f^{\mu} = \left(\frac{\boldsymbol{E}.\boldsymbol{J}}{c}, \rho_e\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B}\right) = j_{\alpha}F^{\mu\alpha}$ | Lorentz force density 4-vector |

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