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# Study of Earth's Obliquity Evolutionary History by Advanced Kinematic Model

**Bijay Kumar Sharma***

*Electronics and Communication Engineering Department, National Institute of Technology, Patna 800005, India*

***Corresponding Author**
Bijay Kumar Sharma, Electronics and Communication Engineering Department, National Institute of Technology, Patna 800005, India.

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## Abstract

*NASA's press release of 'Moon having receded by 1 m from Earth in a quarter of century' on the Silver Jubilee Anniversary (20 July 1994) of Man's landing on Moon led to the development of the Kinematic Model (KM) of evolving Earth-Moon System (E-M system). Best fit KM parameters is adopted for the analysis of evolving E-M system assuming the present length of day of Earth to be 24 hours, sidereal orbital period of Moon divided by spin period of Earth = LOM(length of month)/LOD(length of day) = 27.322 and the age of Moon=4.467Gy. Using the best fit model parameters the velocity of recession of Moon is derived as 2.3cm/y as compared to 3.7cm/y by Lunar Laser Ranging experiment (Dickey et.al. 1994) [1]. Matija Cuk et.al (2016) have proposed a new model for the birth and tidal evolution of our natural satellite Moon born from impact generated terrestrial debris in the equatorial plane of high obliquity, high angular momentum Earth [2]. Using Moon's orbital plane inclination angle (α), Moon's obliquity (β) and eccentricity(e) of Moon's orbit around Earth the advanced KM was developed and used for predicting the evolutionary history of Earth's obliquity from a (semi-major axis of Moon) = $45R_E = 2.96695\times10^8$m (Cassini State Transition orbit) to the present $a = 60.336R_E = 3.844\times10^8$m [3]. The present paper is sequel to the paper Sharma (2024) [5]. In the Advanced KM. (Φ)-Earth's obliquity, (α)-Lunar orbital plane inclination, (β)-Lunar Obliquity and (e)-eccentricity of Moon's orbit around Earth have been included. It has been shown that with the present epoch value of Φ = 23.4°, α = 5.14° , β = 1.54° and e = 0.0549, the general equation describing the evolution of E-M system from $45R_E=2.96695\times10^8$m to the present $a = 60.336R_E = 3.844\times10^8$m yields (Length of Month/Length of Day) = 27.32 with fine tuning of globe-orbit parameters within its measurement error margin. But the tension between the predicted and the observed velocity of recession could not be removed even in the Advanced KM. This tension constrains Moon to be in an accelerated expanding spiral orbit where it recedes from $3.274\times10^8$m to the present orbit $3.844\times10^8$ m in 1.2Gy time span. In the classical model this spiral expansion took 1.9Gy where it monotonically expanded spirally from 18,000Km to 384,440Km.. In the new hypothesis according to Cuk et.al. of Moon's origin from high obliquity, high angular momentum Earth, Moon initially spirals out in chaotic manner with the lunar expansion getting stalled at Laplace Plane Transition as well as in Cassini State transition [2]. Plus in multiple-impact scenario of Moon formation according to Rufu et.al. also results in a delayed formation of full size Moon [4]. Both these factors result in a longer transit time from 18,000Km to $3.012\times10^8$m of 3.267Gy. This results in an accelerated phase from $3.012\times10^8$m to $3.844\times10^8$m in 1.2Gy subsequently resulting into radial recession rate of 3.7cm/y. In the Advanced KM, 3.267Gy is spent in spiral orbital expansion from 3RE to 45RE but in fits and 1.2Gy is spent in accelerated monotonic spiral expansion from $45R_E$ to $60.336R_E$. Hence this model is called fits and bound model. This also results in a much better match between observed LOD curve and theoretical LOD curve as is shown in a sequel paper [5]. In the present paper Advanced Kinematic Model analyzes the history of the evolution of Earth's Obliqquity from the birth of Moon 4.467Gy ago to the present modern times.*

**Keywords:** Earth's Obliquity, Lunar Obliquity, Geo-Synchronous Orbit, Radial Recession, LOM/LOD, Eccentricity, Inclination, Age of Moon

## 1. Introduction

In 1994 at the Silver Jubilee Anniversary of Man's landing on Moon, NASA issued a Press Release stating that Moon had receded by 1 meter in a quarter of a century from 20th July 1969 to 20th July 1994. This enabled the Author to first calculate the length of day curve and compare it with observed LOD curve [6]. This analysis was revised and presented as "Lengthening of Day curve could be experiencing chaotic fluctuations with implications for Earth-Quake Prediction" at the World Space Congrss -2002, Houston, Texas, USA [7]. In the classical Earth-Moon Giant impact scenario, through a glancing angle impact of newly-formed Earth by a Mars sized planetesimal a circumferential disc of impact generated material is created which is a mix of the impactor material(Mars-sized planetesimal) and the target material (newly formed Earth). Beyond Roche's Limit = $3R_E$ = 18,000Km which is greater than first geo-synchronous orbit $a_{G1}$ = 15,000Km, full size Moon accretes and is catapulted by gravitational sling shot on a monotonically expanding spiral orbit as shown in Figure 1 [8].

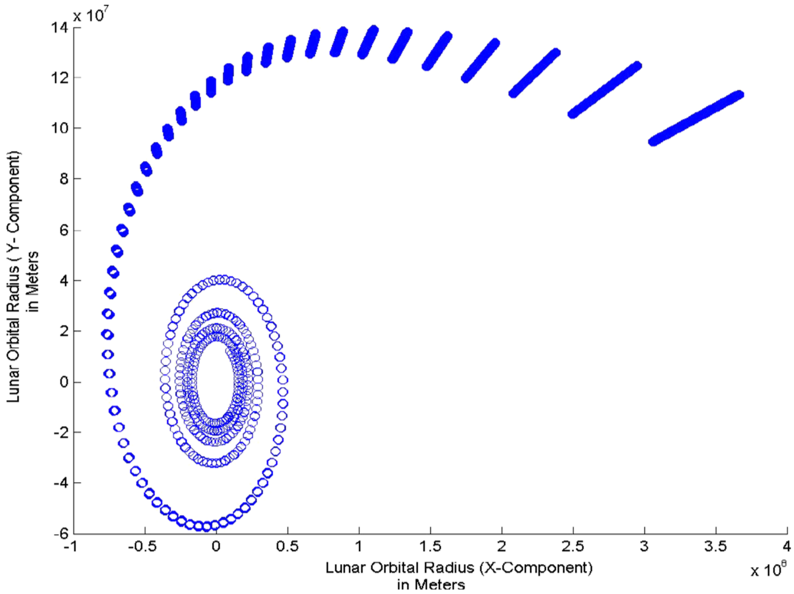


**Figure 1: Lunar Orbital Radius outward expanding spiral trajectory obtained from the simulation for the age of Moon (i.e. from the time of Giant Impact to the present times covering a time span of 4.53Gyrs) [8].**

This model doesnot give the match between Observed and Theoretical Length of Day Curve [5]. In 2016 Matija Cuk et.al. (2016) and his colleagues gave the 'fits and bound' model of Earth-Moon system. According to this model, Moon did not recede from Earth in a monotonically expanding orbit as shown in Figure 1. Our Moon while receding from Earth was interrupted by Laplace Plane Transition and Cassini state transition as shown in Figure A. Cuk et. Al. present a tidal evolution model starting with the Moon in an equatorial orbit around an initially fast-spinning, high-obliquity Earth, which is a probable outcome of giant impacts [2]. Using numerical modelling, they show that the solar perturbations on the Moon's orbit naturally induce a large lunar inclination and remove angular momentum from the Earth–Moon system. The tidal evolution model supports recent high-angular-momentum, giant-impact scenarios to explain the Moon's isotopic composition and provides a new pathway to reach modern Earth's climatically favourable low obliquity of 23.45°.

Cuk and his oolleagues carry out Numerical simulation of the Moon's early tidal evolution from Earth with initial obliquity of 70° and spin period of 2.5h. The result of this simulation is shown in Figure 2.

In the Reference Animation, blue arrow is Earth's spin axis and points to North.
Red is Moon's orbital plane.

The simulation covers a time span of 60My and Moon's spiral orbit expands in fits from $10R_E$ to $18R_E$. There is monotonic expansion from $18R_E$ to $23R_E$.

As animation progresses, Laplace Plane shifts from Earth's equatorial plane to Solar System's ecliptic plane. At $17R_E$ the shift occurs. This is nown as 'Laplace Plane Transition'.

For large Earth's obliquity greater than 68.9°, the Laplace Plane transition causes lunar orbital plane instabilities, Moon's orbit acquires substantial eccentricity and orbital plane inclination driven by solar secular perturbation that operate at high inclination as seen in Kozai resonance Kozai resonance [10].

| Time (My) | Earth's Spin(h) | Ecliptic Component Of J(%) | Laplace Plane | Earth's obliquity (Φ) | Moon's orbital plane inclination (α) | Comment |
|---|---|---|---|---|---|---|
| 0.25 | 2.86 | 153.1 | Equatorial plane of Earth | 70° | 0° | |
| 0.99 | 2.95 | 151.8 | Equatorial plane of Earth | 60° | 0° | |
| 3.4 | 3.21 | 131.8 | Equatorial plane of Earth | 55° | oscillatory | Falls into resonance |
| 9.5 | 3.73 | 127.2 | Equatorial plane of Earth | 50° | oscillatory | Moon moves out and gets stuck in evection resonance |
| 12 | 3.98 | 139 | Transition | 40° | oscillatory | Unstable |
| 20.5 | 5.66 | 112.1 | Ecliptic | 30° | oscillatory | Comes out of resonance |
| 25 | | | | | | Stalled |
| 28.6 | 6.4 | 102.8 | Ecliptic | 20° | oscillatory | Stalled |
| 40 | | | | | | Stalled |
| 44.8 | 7.04 | 103.6 | Ecliptic | 15° | 20° | Expansion resumed |
| 59.5 | 7.35 | 103.1 | Ecliptic | 10° | 25° | expanding |

**Table 1: The history of Earth's obliquity (Φ) and Moon's orbital plane inclination (α) during Laplace Plane transition as concluded by numerical simulation.**

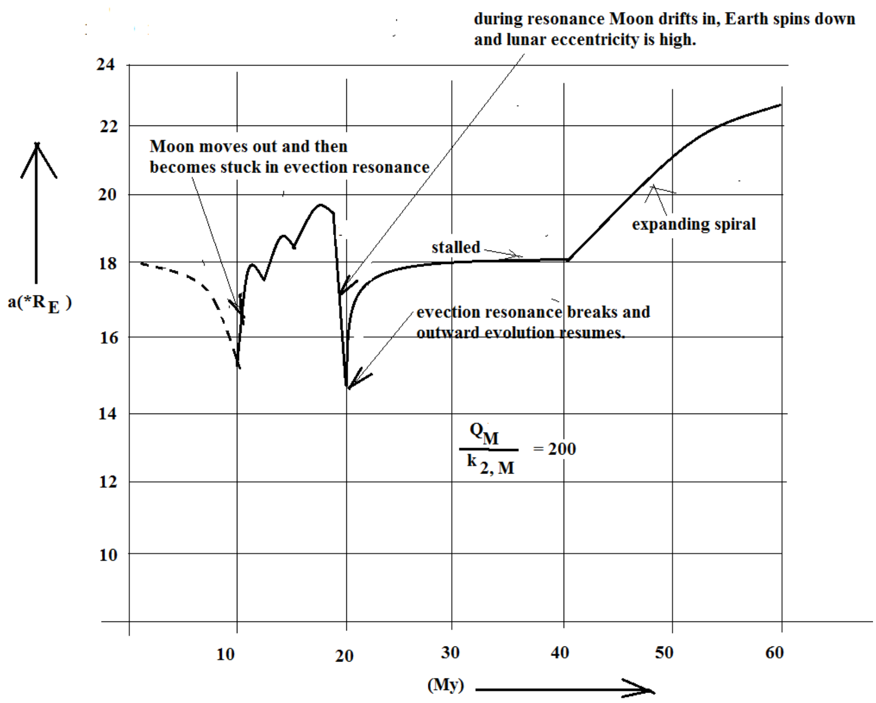


**Figure 2: Evolution of semi-major axis of Moon for first 60My when Laplace plane transition is encountered at 20My and a = 17$R_E$ Matija Cuk et al (2016) [3].**

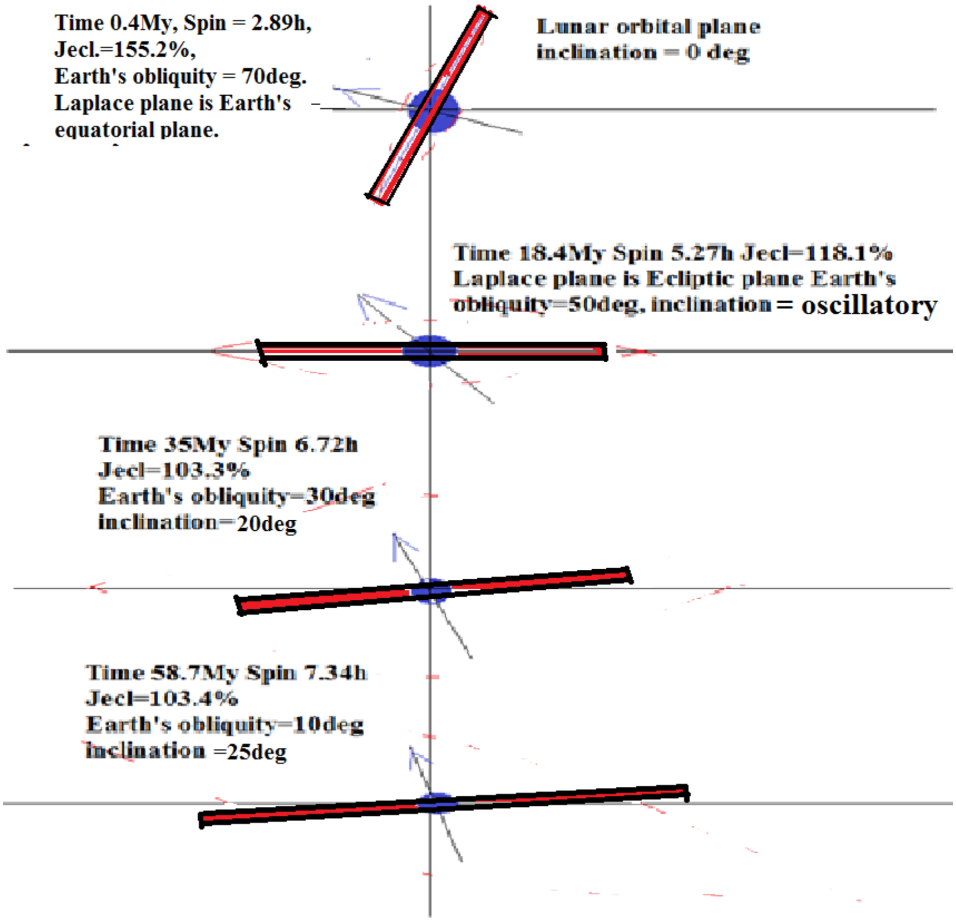


**Figure 3: ANIMATION:E-M configuration in four time slots:0.4My, 18.4My, 35My and 58.7My. tilted Earth has become vertical and Moon's orbital plane has acquired inclination in Laplace Plane transition Matija Cuk et al (2016) [3].**

For any perturbed orbit, there exists a Laplace Plane around the normal of which the normal of the orbital plane of the perturbed orbit processes [9]. The Laplace Plane undergoes a transition during lunar tidal evolution when the Moon recedes from inner region dominated by perturbation of the Earth's equatorial bulge to the region dominated by solar perturbation.

$$r_L(laplace\ plane\ transition\ orbit\ radius) = (2J_2 \frac{M_E}{M_S} R_E^2 a_E^3 \times (1-e^2)^{3/2})^{1/5} \qquad A$$

$J_2$ = oblateness moment of Earth. As Earth spin slows down, oblateness decreases leading to $r_L$ moving inward.

$$r_L = 16 \sim 22R_E \qquad B$$

For Earth's obliquity $\Phi < 68.9°$ , Laplace Plane transition is smooth and inclination and eccentricity remain zero. But for Earth's obliquity $\Phi > 68.9°$ , Laplace Plane transition causes orbital instability, acquire substantial eccentricity and inclination driven by solar secular perturbation that operate at high inclination as seen in Kozai resonance [10].

Tidal evolution of the Moon from high obliquity Earth followed by inclination damping at the Cassini state transition [10,11].

Tides raised on the Earth by the Moon have caused an expansion of the spiral orbit [11]. Tides raised on Moon by Earth have de-spun the Moon to synchronous rotation and driven the spin axis of Moon to a Cassini State. Cassini State is a coprecessing configuration Lunar's spin axis coplanar with the lunar orbit normal and with the normal of the Laplace plane (which at present is coincident with the normal of the ecliptic).

Moon is pushed-out due to Earth's tides but is pushed-in due to Moon's tide.

### 1.1 Cassini State Transition

After Laplace Plane transition, Moon continues to recede and lunar rotation passes through Cassini state transition. Regardless of the nature of the lunar rotation state, Moon's obliquity is very high during Cassini state transition and immediately following it, leading to the damping of lunar plane inclination. Lunar plane inclination is damped from 30° ( obtained during Laplace plane transition) to the present value of 5.334° if we assume the long-term average tidal properties for Earth and non-dissipative Moon in synchronous rotation states.

Animation of relative orientation of lunar figure (lunar orbital inclination and lunar spin axis obliquity) during Cassini state transition following the simulation plotted in Figure 5. In the Cassini-state transition animation, the Moon is seen from the ascending node of lunar orbit with the ecliptic plane (i.e. the Moon's Laplace plane at the time) parallel to the horizontal axis. The red arrow shows the orientation of the Moon's orbit normal. At first the Moon's orbit normal and Moon's spin axis are on the same side of the normal to the Ecliptic, indicating that the Moon is in Cassini State 1.Once Cassini state is de-stabilized after some wobbling, the Moon settles in non-synchronous state somewhat similar to Cassini State 2 (with orbit normal and spin axis being on the opposite sides of the normal to the ecliptic). During this time both the inclination and obliquity (which is forced by inclination) are being damped by lunar obliquity tides. At a = 35.1$R_E$ the Moon becomes synchronous again and enters Cassini State 2, where it stays for the rest of the simulation (this event is visible as 5° jump in obliquity).

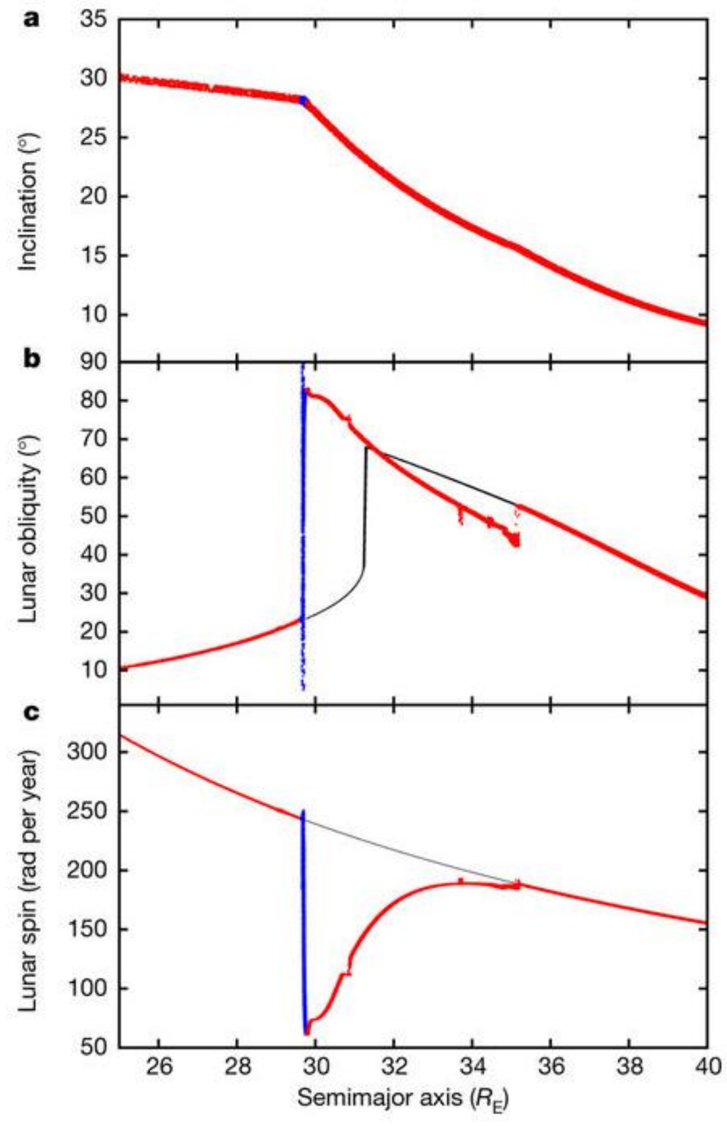


**Figure 4: Numerical integration of the later phase of lunar tidal evolution assuming a lunar inclination of 30° at 25$R_E$ and the current tri-axial shape of Moon Matija Cuk et al (2016) [3].**

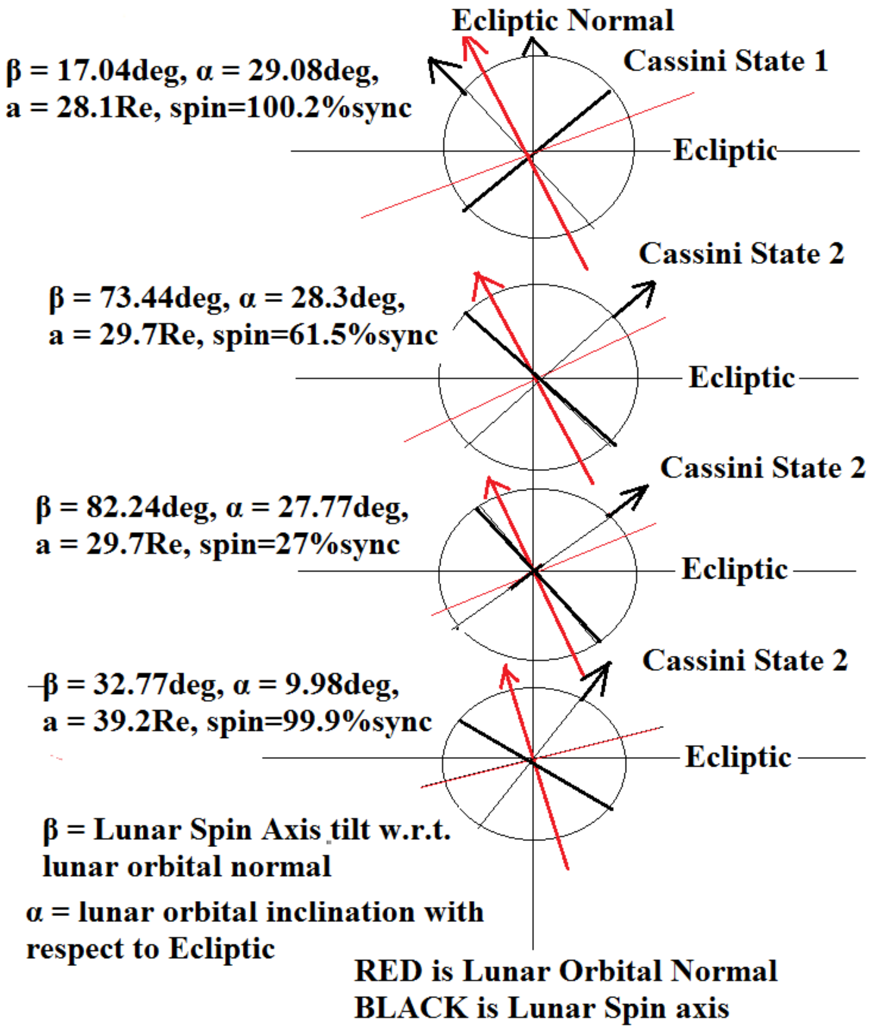


**Figure 5: Animations at different stages of Cassini State Transition Matija Cuk et al (2016) [3].**

Since the current triaxial shape of the Moon matches the order of magnitude of tidal deformation expected at 23 to 26$R_E$, it was assumed that Moon is rigid and has present triaxial shape frozen at 26RE.

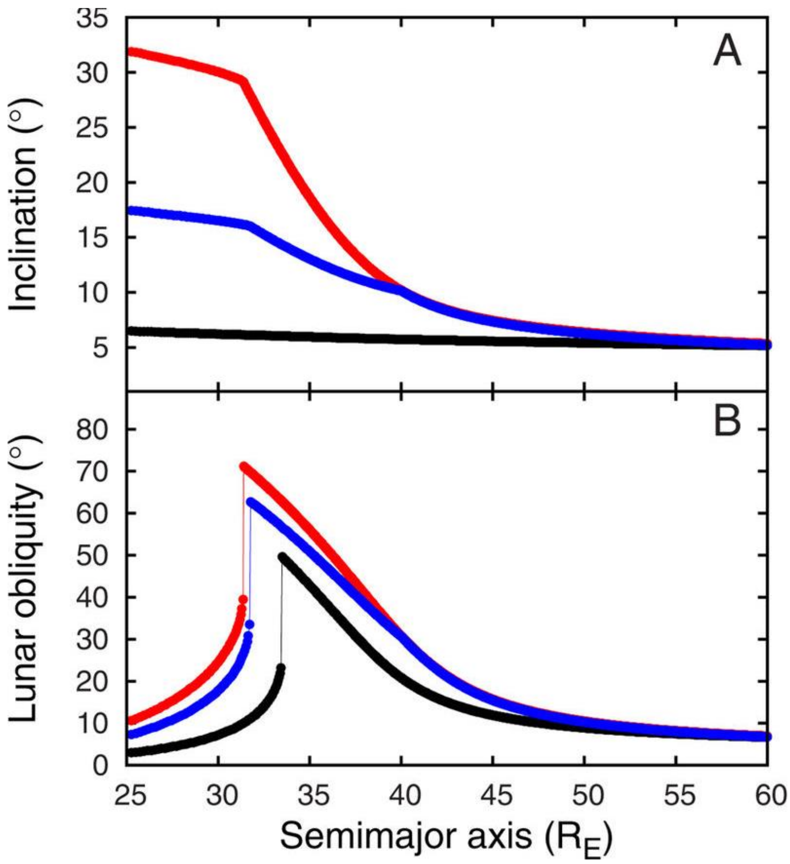


**Figure 6: Black lines have $Q_M$ = 10,000 and Red lines have $Q_M$ = 38 (current value of Moon)Matija Cuk et al (2016) [3].**

**$Q_E$ = 33 ~ 35 so that the evolution from $25R_E$ to $60R_E$ is covered in 4.5Gy.**

**Blue lines have $Q_M$ = 100 for a < $40R_E$ and $Q_M$ = 38 for a > $40R_E$.**

Black lines closely resembles the results of studies that neglected lunar obliquity tides while the other two curves indicate that the past lunar inclination must have been much larger owing to lunar obliquity tides.

**Jumps between $30R_E$ and $35R_E$ is due to transitions between Cassini states 1 and 2 [18]**

Principal moment of inertia beyond $25R_E$ is the Principal moment of inertia of the current lunar shape frozen at a distance of 15 to 17RE in an orbit with e~ 0.2

J (initial) = 1.8J0 and Earth's obliquity = 70°.

Solar perturbations induce sizeable lunar eccentricity as Moon reaches rL = $17R_E$ triggering strong eccentricity damping lunar tides that shrink the semi-major axis and approximately balance the outward push by Earth's tides.

Eccentric orbits remove AM from the lunar orbit and transfer it to Earth's heliocentric orbit. Earth tides in turn transfer AM from Earth's spin to lunar orbit. During this prolonged stalling of lunar tidal evolution Moon acquires large inclination (over 30°) while Earth's obliquity decreases.

Laplace Plane transition leads to large AM loss, high lunar inclinations and low terrestrial obliquity. Earth's obliquity = 80° experiences stalling and reversal of tidal evolution with Moon getting caught in a death spiral. Where does the Moon's orbital plane inclination come into picture if initially Moon was formed from the accretion of impact debris in Earth's equatorial plane? Today the inclination should have been zero but it is not zero. Through AKM evolutionary history of Earth's Obliquity, Moon's Obliquity, Moon's orbital plane inclination and lunar orbit's eccentricity are studied from the time of birth of Moon to the modern times.

Method:

### 1.2 Determination of the Total Angular Momentum of E-M System and its Orientation in the Present epoch.

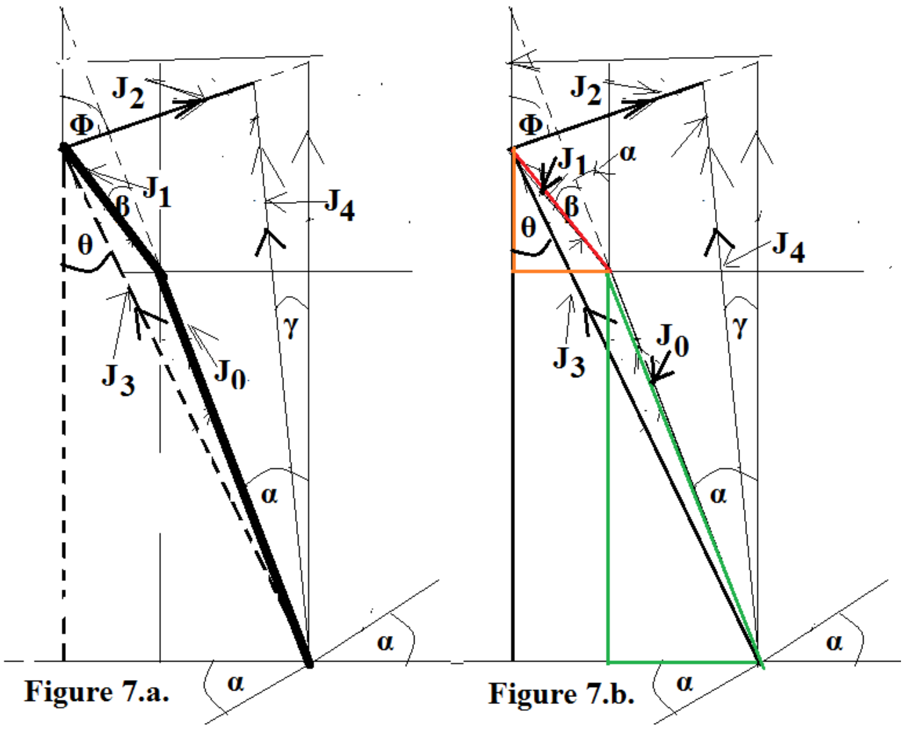


**Figure 7: Vector Diagram for calculating the resultant angular momentum of Earth-Moon system. [Courtesy: Author]**

Moon's orbital plane inclination at present with respect to (w.r.t.) the ecliptic normal = α = 5.14° = 0.08970992355250854 radians;
Moon's spin axis obliquity with respect to (w.r.t.) the lunar orbital plane = β ( Moon's axial tilt ) = 1.54° = 0.026878070480712675 radians ;
Earth's spin axis obliquity with respect to (w.r.t.) the ecliptic normal = Φ = 23.44° = 0.40910517666747087 radians ;

$$\overrightarrow{J_E} = \lfloor J_E \rfloor\ 23.44° \quad to\ the\ right\ of\ the\ ecliptic\ normal$$

$$J_E = \boldsymbol{J_2} = C \times \frac{2\pi}{T_{spin\_Earth}} = 0.3308 \times M_{Earth} \times R_{Earth}^2 \times \frac{2\pi}{23.9345 \times 3600}$$

$$= 5.84758 \times 10^{33} \frac{Kg - m^2}{s} \qquad 1$$

$$where\ Sidereal\ spin\ period = 23.9345h,$$

$$Volumetric\ mean\ radius\ of\ Earth = 6371.008Km, \qquad M_E = 5.9723 \times 10^{24} Kg;$$

Substituting the magnitudes of the parameters we get:

$$C = 8.01906 \times 10^{37} Kg - m^2\ and\ \omega = 7.29211 \times 10^{-5} radians\ per\ s.$$

$$J_{ORB} = \boldsymbol{J_0} = \frac{m}{1 + \frac{m}{M}} \times a^2 \times \frac{2\pi}{T_{ORB}} \sqrt{1 - e^2}$$

$$= 1.07066 \times 10^{40} Kg - m^2 \times \frac{2.6617 \times 10^{-6} rad}{s} = 2.84978 \times 10^{34} \frac{Kg - m^2}{s} \qquad 2$$

Here 'a' (semi-major axis of Moon's orbit) = 3.844×10⁸m; and mass of our Moon m = 0.07346×10²⁴ Kg, m/(1+m/M) = reduced mass =7.25674×10²² Kg, TORB orbital period of Moon around Earth(sidereal month) = 27.3217d and e is eccentricity = 0.0549.

$$\overrightarrow{J_M} = \overrightarrow{\boldsymbol{J_1}} = \ Moon's\ spin\ angular\ momentum\ in\ the\ direction\ of\ Moon'spin\ axis$$

As seen in Figure 1 , Moon's spin axis is tilted to Ecliptic normal by 6.68° to the left of Ecliptic normal.

$$I = 0.394 \times m \times R_{Moon}^2 = 8.73669 \times 10^{34} Kg - m^2, \Omega = 2.6617 \times 10^{-6} radians/s$$

Hence

$$\overrightarrow{J_M} = \ \lfloor J_M \rfloor\, 6.68° \qquad 3$$

$$J_M = \boldsymbol{J_1} = I \times \Omega = 2.32541 \times 10^{29} \frac{Kg - m^2}{s} \qquad 4$$

As seen in Figure 1:

$$\overline{J_3} = \overline{J_0} + \overline{J_1}\ and\ \overline{J_4} = \overline{J_2} + \overline{J_3}$$

As already defined:

$$J_0 = lunar\ orbital\ J\ , J_1 = Moon's\ spin\ J, J_2 = Earth'sspin\ J$$

$$and\ J_4 = vectorial\ resultant\ total\ J\ of\ E - M\ system$$

$$J_3 = 2.84980032457008363 \times 10^{34} \frac{Kg - m^2}{s} \qquad 5$$

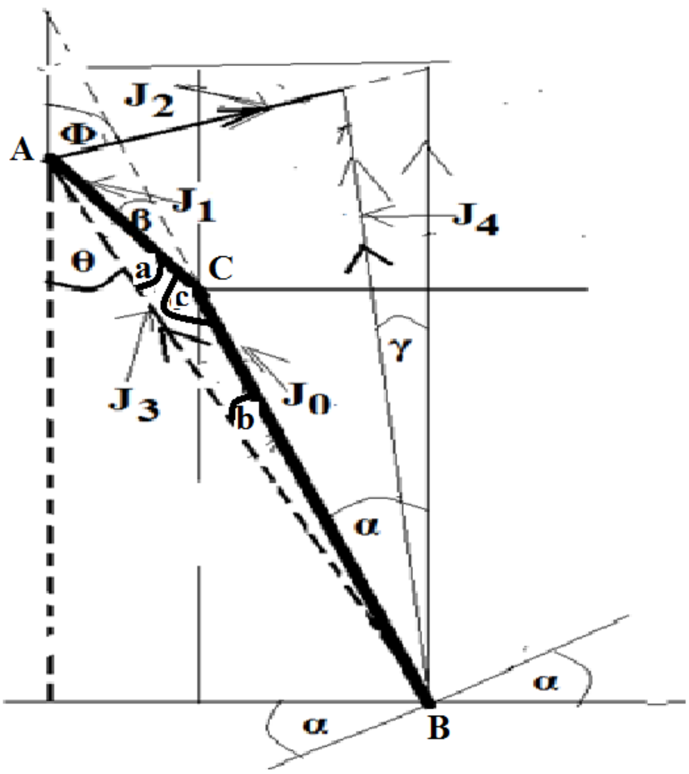


**Figure 8: Triangle ABC bounded with $J_0$, $J_1$, and $J_3$ with angles a, b and c opposite sides $J_0$ , $J_1$ , and $J_3$. [Courtesy: Author]**

Inspecting Figure 2, we see that $J_3$ makes an angle (θ) w.r.t. the normal of the ecliptic.

$$\theta = ArcTan = 0.8971014284837965\ radians = 5.14001256472789° \quad 6$$

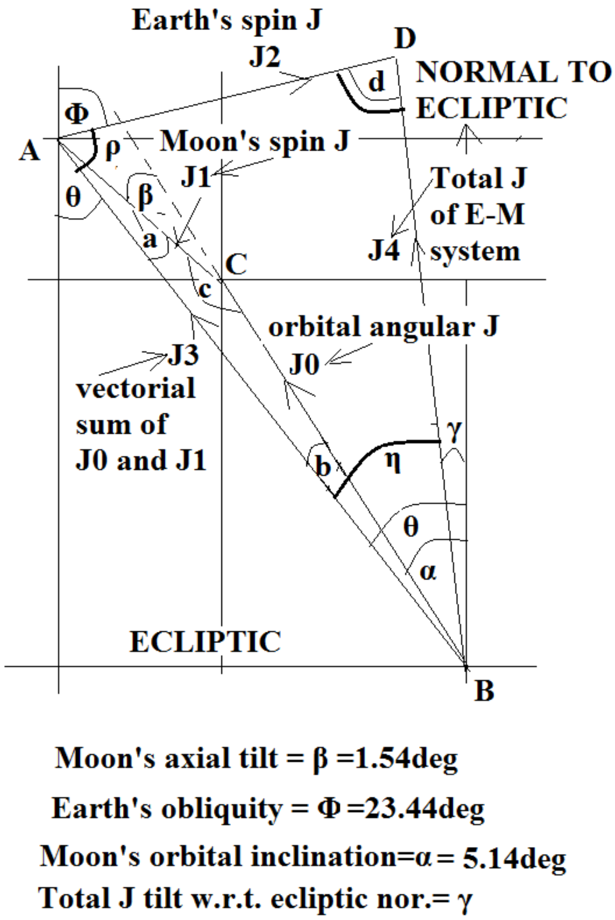


**Figure 9: Two triangles of J vectors: Triangle ABC sides J0(orb. J), J1 (Moon's spin J) and J3 where J3 is the vector sum of J0 and J1. [Courtesy: Author]**

Triangle ABD sides J2 (Earth's spin J), J3(the vector sum of J0 and J1) and J4(total J of E-M system)

$$J_4 = 3.3749210029333725 \times 10^{34} \frac{Kg - m^2}{s} \quad 7$$

The total angular momentum $J_4$ is inclined w.r.t. ecliptic normal at γ where

$$\gamma = \theta - \widetilde{b}\,(vertice\ angle\ of\ \Delta ABD)$$

$$= 0.006727272788964655 99\ radians = 0.385445° \quad 8$$

## 2. Validation of Present epoch LOM/LOD= 27.32 by K.M

In CELMEC paper CELE-D-17-00144, it has been proved that:

$$(N)^2 \times a^3 = X^2 + (F)^2 \times (a^2)^2 + G^2 +$$

$$2(F \times a^2)(G)\sqrt{1 - D^2} - 2 \times X$$

$$\times \sqrt{(F \times a^2)^2 + (G)^2 + 2\,(F \times a^2)(G)\sqrt{1 - D^2}} \times \{Z\} \quad 9$$

Where:

$$\frac{J_4}{C \times B} = N, \qquad B = \sqrt{GM + Gm} = 2.00873 \times 10^7 \frac{m^{3/2}}{s}$$

$$\frac{F^*}{C} = F \; and \; \frac{I}{C} = G \; and \; I = 0.394 \times m \times R_{Moon}^2 = 8.73669 \times 10^{34} Kg - m^2$$

$$C = 0.3308 \times M_{Earth} \times R_{Earth}^2 = 8.01906 \times 10^{37} Kg - m^2$$

$$N = 2.09707715634 \times 10^{-11} \left(\frac{1}{m^{\frac{3}{2}}}\right), G = 0.00108949,$$

$$F^* = \frac{m}{1 + \frac{m}{M}} \times \sqrt{1 - e^2} \;, F = \frac{m^*}{C} \; with \; \sqrt{1 - e^2} excluded$$

$$hence \; F = 9.04936 \times 10^{-16} \left(\frac{1}{m^2}\right)$$

Let Sin[β] = D and Cos[β] = √(1-$D^2$),
Sin[α] = A and Cos[α] = √(1-$A^2$),
Sin[Φ] = B and Cos[Φ] =√(1-$B^2$),

Let Z= - Cos[α] Cos[Φ]+ Sin[α] Sin[Φ]=A.B-√(1-$A^2$) √(1-$B^2$) 10

Using current Earth's obliquity (Φ = 23.44°), current Moon's orbital inclination (α = 5.14°) and current Moon's obliquity (β=1.54°)

$$Z = \left\{-\sqrt{1 - A^2}\sqrt{1 - B^2} + A.B\right\} = -0.87815$$

And Cos[β] =√(1-$D^2$) = 0.999639 ; D = 0.0268748;
The value of N has been obtained after some fine tuning so that (9) is satisfied for the current values of ω/Ω (LOM/lOD), α (Inclination angle) , β (lunar obliquity), Φ (terrestrial obliquity) and e (eccentricity).
Substituting (10) in (9) we get:

$$-7152.75 + 234.494.X + X^2 \; == 0 \qquad 11$$

This is a quadratic equation and its roots are:

$$\frac{\omega}{\Omega} = \frac{LOM}{LOD} = 27.32 \; or - 261.814 \qquad 12$$

The fine tuning of 'N' has been done within the margin of measurement error of globe parameters and orbital parameters so that (9) yields the present epoch LOM/LOD = 27.32.

### 3. Evolutionary Spatial Functions of Terrestrial Obliquity(Φ) and LOM/LOD

Evolutionary spatial functions of inclination angle (α), Moon's obliquity(β) and of eccentricity 'e' have been determined in CELE-D-17-00144. They are as follows:

$$Inclination \; angle \; \alpha = \frac{1.18751 \times 10^{25}}{a^3} - \frac{7.1812 \times 10^{16}}{a^2} + \frac{1.44103 \times 10^8}{a} - 8.250567342 \times 10^{-3} \qquad 13$$

$$Moon's \; Obliquity \; angle \; \beta = 3.36402 - 1.37638 \times 10^{-8} a + 1.32216 \times 10^{-17} a^2 \qquad 14$$

$$e = 0.210252 + 8.38285 \times 10^{-10} a - 3.23212 \times 10^{-18} a^2 \qquad 15$$

The LOM/LOD and Earth's obliquity angles are tabulated in Table 1.

| a(×$R_E$) | a(×$10^8$m) | LOM/LOD | Φ (radians) | Φ° |
|---|---|---|---|---|
| 30 | 1.9113 | 23.3752 | unstable | unstable |
| 35 | 2.22985 | 26.1194 | unstable | unstable |
| 40 | 2.5484 | 28.1147 | unstable | unstable |
| 45 | 2.86695 | 29.2938 | 0.113792 | 6.51 |
| 50 | 3.1855 | 29.5965 | 0.220227 | 12.6 |
| 55 | 3.50405 | 28.9877 | 0.314929 | 18 |
| 60 | 3.8226 | 27.4 | 0.398676 | 22.84 |
| 60.335897 | 3.844 | 27.32 | 0.409105 | 23.44 |

**Table 2: LOM/LOD and Earth's Obliquity for past geological epochs**

### 3.1 Evolutionary Function of LOM/LOD

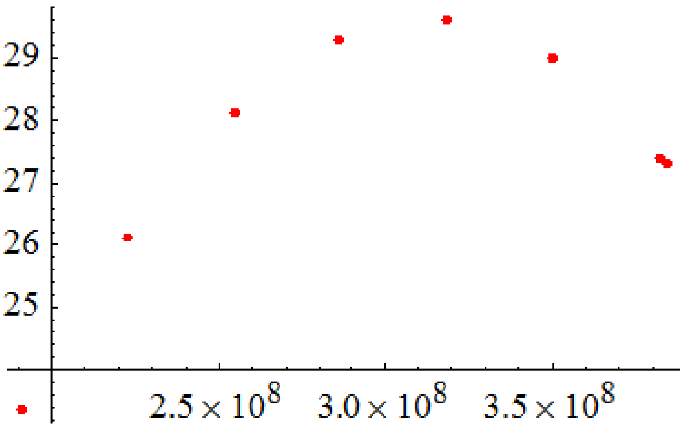


**Figure 10: ListPlot of LOM/LOD in different geologic epochs as given in Table 1. [Courtesy: Author]**

$$\frac{LOM}{LOD} = \frac{\omega}{\Omega} = -12.0501 + 2.6677 \times 10^{-7} \times a - 4.27538 \times 10^{-16} \times a^2 \qquad 16$$

The plot of (16) is as follows:

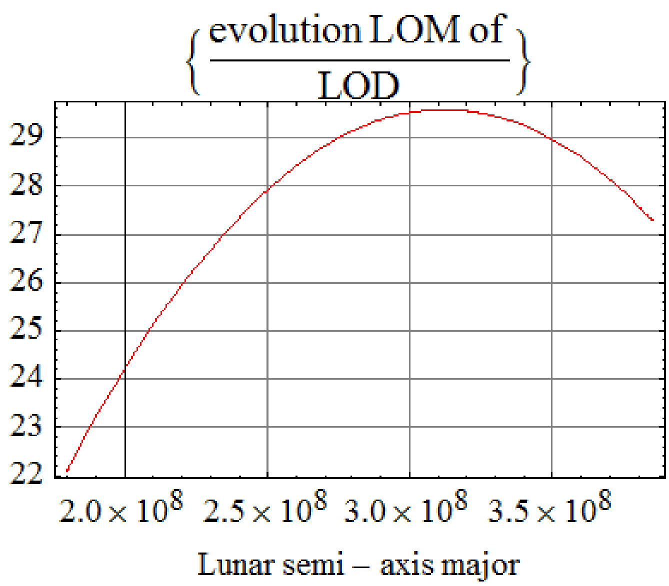


**Figure 11: Plot of FIT function given by (16). [Courtesy: Author]**

Superposition of LOM/LOD ListPlot and Fit Plot is given in Figure 3.

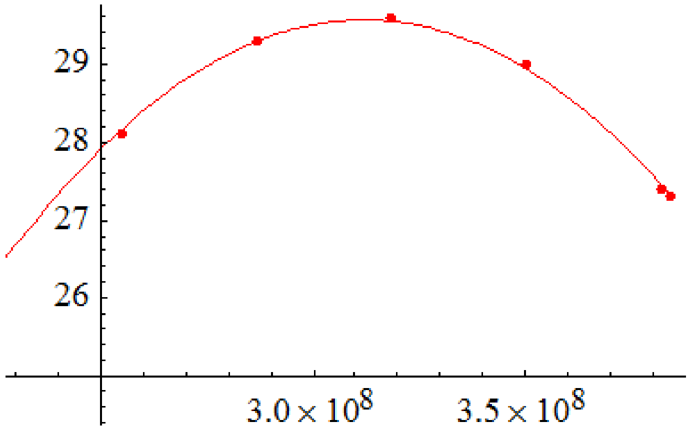


**Figure 12: Superposition of LOM/LOD ListPlot and Fit Plot. [Courtesy: Author]**

The correspondence between LISTPLOT and FIT PLOT is good hence (16) gives the evolutionary history of LOM/LOD.

### 3.2. Evolutionary Function of Earth's Obliquity

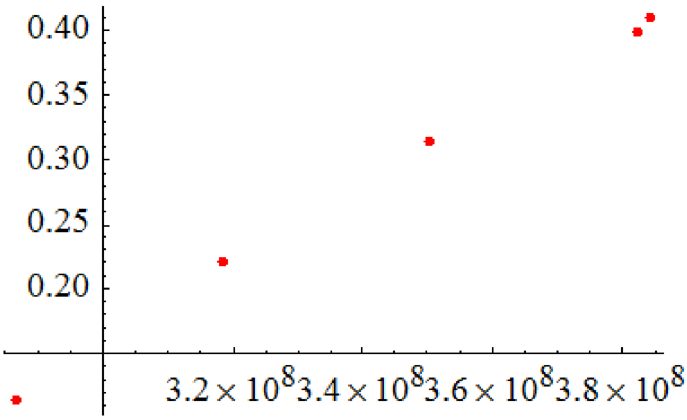


**Figure 13: List Plot of Earth's obliquity (Φ) angle over different geological epochs. [Courtesy: Author]**

The approximate FIT function to the ListPlot of Earth's obliquity in Figure 4 is:

$$\varphi = -0.732299 + 2.97166 \times 10^{-9} \times a \qquad 17$$

The Plot of (17) is as follows:

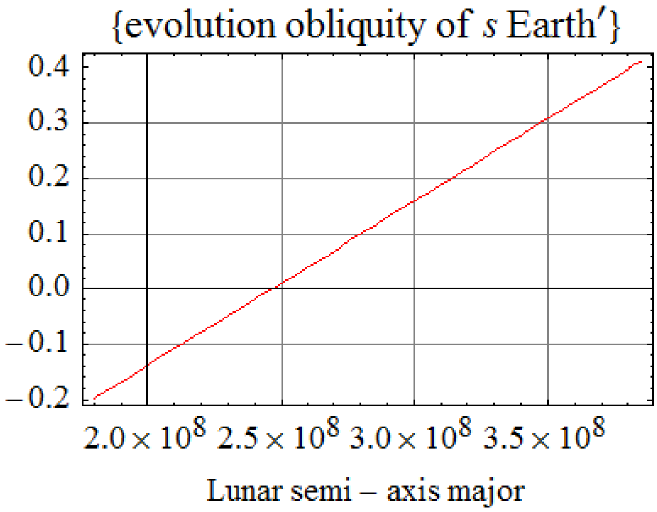


**Figure 14: Plot of FIT function given by (17). [Courtesy: Author]**

Superposition of ListPlot and Fit function is given in Figure 6.

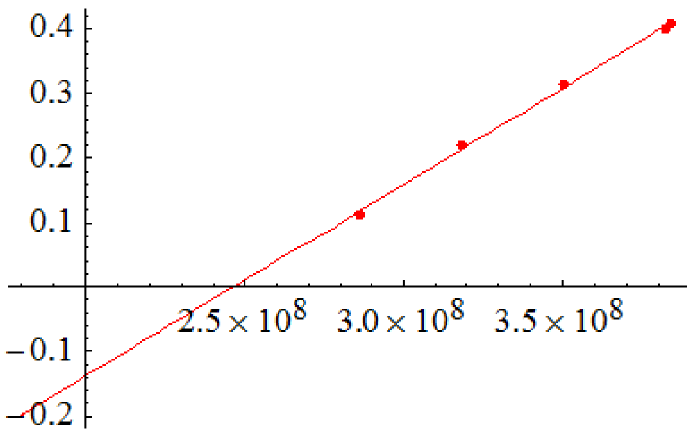


**Figure 15: Superposition of ListPlot of Earth's obliquity (Φ) and Fit Plot. [Courtesy: Author]**

The correspondence between LISTPLOT and FIT PLOT is good hence (17) gives an accurate evolutionary history of Earth's obliquity.

We have altogether 5 spatial function (13), (14), (15), (16) and (17) describing the evolution of inclination angle (α), Moon's obliquity (β), eccentricity(e) of lunar orbit, LOM/LOD and Earth's obliquity (Φ) respectively through different geologic epochs.

| a (×$R_E$) | a (×$10^8$m) | ω/Ω | α radians | β | e | Φ(rad) | Sin[Φ] |
|---|---|---|---|---|---|---|---|
| 30 | 1.9113 | 23.3752 | 0.480685 (27.4°) | 1.21635 (69.69°) | 0.2524 | unstable | -0.464076 |
| 35 | 2.22985 | 26.1194 | 0.26478 (15.17°) | 0.952317 (54.56°) | 0.236 | unstable | -0.216896 |
| 40 | 2.5484 | 28.1147 | 0.168969 (9.68°) | 0.71512 (40.97°) | 0.214 | unstable | -0.0195376 |
| 45 | 2.86695 | 29.2938 | 0.124631 (7.1408°) | 0.504756 (28.92°) | 0.1849 | 0.113792 (6.51°) | 0.113547 |
| 50 | 3.1855 | 29.5965 | 0.103801 (5.04736°) | 0.321225 (18.4°) | 0.1493 | 0.220227 (12.6°) | 0.218451 |
| 55 | 3.50405 | 28.9877 | 0.0941394 (5.39379°) | 0.164527 (9.4267°) | 0.10714 | 0.314929 (18°) | 0.309749 |
| 60 | 3.8226 | 27.4 | 0.0898729 (5.149°) | 0.03466 (1.986°) | 0.0584 | 0.398676 (22.84°) | 0.388198 |
| 60.336 | 3.844 | 27.32 | 0.08971 (5.14°) | 0.0268 (1.54°) | 0.0549 | 0.409105 (23.44°) | 0.397788 |

**Table 3: Evolutionary History of ω/Ω (LOM/lOD),α (Inclination angle) , β (lunar obliquity), e (eccentricity) and Φ (terrestrial obliquity).**

## 4. Validation of Radial Recession Velocity of Moon = 3.7cm/y

The Tidal Torque of Satellite on the Planet and of Planet on the Satellite = Rate of change of angular momentum hence

$$Tidal\ Torque = T = \frac{dJ_{orb}}{dt} \qquad 18$$

But Orbital Angular Momentum using Kepler's Third Law:

$$J_{orb} = m^* a^2 \times \frac{B}{a^{3/2}} = m^* B\sqrt{a} \qquad 19$$

Time Derivative of (15) is:

$$T = \frac{dJ_{orb}}{dt} = \frac{m^* B}{2\sqrt{a}} \times \frac{da}{dt} \qquad 20$$

In super-synchronous orbit, the radius vector joining the satellite and the center of the planet is lagging planetary tidal bulge hence the satellite is retarding the planetary spin and the tidal torque is BRAKING TORQUE. This is shown in Figure 7 in context of Earth and Moon.

In sub-synchronous orbit, the radius vector joining the satellite and the center of the planet is leading planetary tidal bulge hence the satellite is spinning up the planet and the tidal torque is ACCELERATING TORQUE.

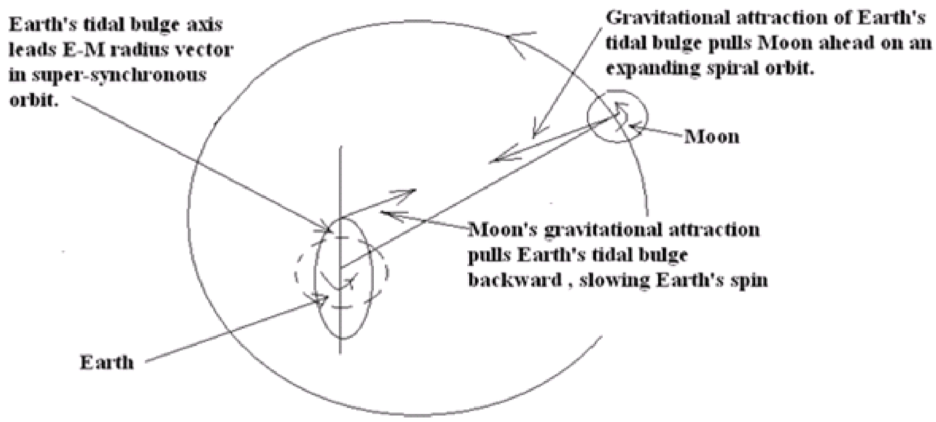


**Figure 16: In Earth-Moon System , Moon is in super-synchronous orbit. The off-setting of the line of bulge in Earth with respect to E-M radius vector creates a tidal drag and de-spinning of Earth leading to secular lengthening of day. The de-spinning of Earth leads to transfer of angular momentum to Moon resulting in Moon's recession. During the conservative phase of the gravitational sling shot impulsive torque, Moon is launched on a expanding spiral path as shown in Figure 1. After the conservative phase,Earth coasts on its own towards the outer Clark's orbit where it will get locked-in in dustant future [Courtesy: Author]**

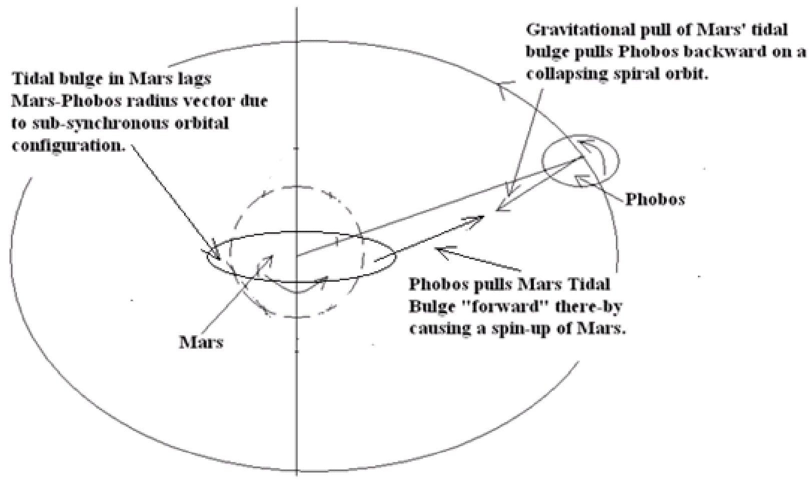


**Figure 17: In Mars-Phobos system, Phobos falls in sub-synchronous orbit and the off-setting of thePhobos bulge with respect to the radius vector joining Mars and Phobos causes a tidal spin-up of Mars. In the process Phobos is launched on a gravitational runaway collapsing orbit also known as death spiral [Courtesy: Author]**

The Author has assumed the empirical form of the Tidal Torque as follows:

$$T = \frac{K}{a^Q}\left[\frac{\omega}{\Omega} - 1\right] \qquad 21$$

(21) implies that at Inner Clarke's Orbit and at Outer Clarke's Orbit where ω/Ω= 1(geo-synchronous orbits), tidal torque is zero and (20) implies that radial velocity is zero and there is no spiral-in or spiral-out.

At Triple Synchrony, Satellite-Planet Radius Vector is aligned with planetary tidal bulge and the system is in equilibrium. But there are two roots of ω/Ω=1: Inner Clarke's Orbit and Outer Clarke's Orbit. As already shown in Total Energy Profile (CELE-D-17-00145), inner Clarke's Orbit aG1 is a maxima hence unstable equilibrium state and Outer Clarke's Orbit aG2 is a minima hence stable equilibrium state. In any Binary System, secondary is conceived at aG1 . This is the CONJECTURE assumed in Kinematic Model. From this point of inception Secondary may either tumble short of aG1 or tumble long of aG1 . If it tumbles short, satellite gets trapped in Death Spiral and it is doomed to its destruction as all hot Jupiters are destined to be. If it tumbles long, satellite gets launched on an expanding spiral orbit due to gravitational sling shot impulsive torque during of conservative phase which quickly decays due to the growing differential of ω and Ω and the resulting tidal heating. After the impulsive torque has decayed, the satellite coasts on it own toward final lock-in at aG2 .

Equating the magnitudes of the torque in (20) and (21) we get:

$$\frac{m^{*}B}{2\sqrt{a}}\times\frac{da}{dt}=\frac{K}{a^{Q}}\left[\frac{\omega}{\Omega}-1\right] \qquad 22$$

Rearranging the terms in (22) we get assuming X= ω/Ω:

$$V(a)=Velocity\ of\ recession=\frac{2K}{m^{*}B}\times\frac{\sqrt{a}}{a^{Q}}[X-1]m/s \qquad 23$$

The Velocity in (23) is given in m/s but we want to work in m/y therefore (23) R.H.S is multiplied by 31.5569088×10⁶s/(solar year).

$$V(a)=\frac{2K}{m^{*}B}\times\frac{\sqrt{a}}{a^{Q}}[X-1]\times 31.5569088\times 10^{6}m/y \qquad 24$$

Equation (9) gives the value of X in advanced Kinematic Model:

$$(N)^{2}\times a^{3}=X^{2}+(F)^{2}\times(a^{2})^{2}+G^{2}+$$

$$2(F\times a^{2})(G)\sqrt{1-D^{2}}-2\times X$$

$$\times\sqrt{(F\times a^{2})^{2}+(G)^{2}+2(F\times a^{2})(G)\sqrt{1-D^{2}}}\times\{Z\} \qquad 9$$

In (24) ‘a’ refers to the semi-major axis of the evolving Satellite. There are two unknowns: exponent ‘Q’ and structure constant ‘K’. Therefore two unequivocal boundary conditions are required for the complete determination of the Velocity of Recession.

If Laplace Plane Transition and Cassini State Transition are neglected then First boundary condition is at $a = a_2$ which is a Gravitational Resonance Point where ω/Ω = 2 [Rubicam 1975],
i.e. $(Na^{3/2} – Fa^{2})= 2$ has a root at $a_2$.
In E-M case, $a_2 = 2.40649\times10^{7}$ m.
At $a_2$ the velocity of recession maxima occurs. i.e. $V(a_2) = V_{max}$.
Therefore at $a = a_2$, $(\delta V(a)/\delta a)(\delta a/\delta t)|_{a2} = 0$.
On carrying out the partial derivative of V(a) with respect to ‘a’ we get the following:

$$At\ a_2,\quad (2-Q)A\times a^{1.5}-(2.5-Q)F\times a^{2}-(0.5-Q)=0 \qquad 25$$

Solving (21) at 2:1 Mean Motion Resonance orbit ‘a2’ we obtain :

$$Q=3.22684 \qquad 26$$

Now structure constant (K) has to be determined . This will be done by trial error so as to get the right age of Moon[1] i.e. 4.46Gy. Rewriting (24) and substituting the best fit values of the exponent and constants N and F we obtain the structure constant ‘K’.

$$V(a)=\frac{2K}{m^{*}B}\times\frac{\sqrt{a}}{a^{Q}}[X-1]\times 31.5569088\times 10^{6}m/y \qquad 24$$

We will assume the age of Moon 4.46Gy as already mentioned in Foot Note 3. The Transit Time from $a_{G_1}$ to the present ‘a’ is given as follows:

$$Transit\ Time=\int_{a_{G1}}^{a}\frac{1}{V(a)}da \qquad 24$$

Since Laplace Plane Transition and Cassini State Transition has to be considered therefore we will consider the time span spent in spiraling out from $2.86695 \times 10^8$m to the present orbit of $3.844 \times 10^8$m.

Paleo-astronomical studies conducted by Wells(1963,1966) , Charles P.Sonnet (1998) and Arbab(2009) tell us that transition from $45R_E = 2.86695 \times 10^8$m to the present has taken 3Gy. So the definite integral in transiting from $2.87 \times 10^8$m to $3.844 \times 10^8$m should give 3Gy.

Using several iterations we will determine the structure constant 'K' which yields transit time from birth to the present to be 3Gy within the limits of definite integral . These values of 'K' and 'Q' will be utilized in the advanced Kinematic Model to determine the radial velocity of recession.

By classical E-M model Q is calculated to be Q = 3.22684.
K = $5.5 \times 10^{42}$Newton/( ),Transit Time (from $3.012 \times 10^8$m to $3.844 \times 10^8$m) =2.38Gy.
This gives present epoch velocity of recession of Moon as =2.4cm/y.

K = $8.33269 \times 10^{42}$Newton-mQ that Moon's crust frmed ,Transit Time (from $3.012 \times 10^8$m to $3.844 \times 10^8$m) = 1.57732 Gy.

This gives present epoch velocity of recession of Moon as = 3.7cm/y.

## 5. Discussion

LLR measurement of 3.7cm/y was resulting in too short an age of Moon (~ 3Gy) which was contrary to the observed age of the rocks brought from Moon during Apollo Missions from 1969 to 1972 (curation/Lunar-NASA). These missions brought 382Kg of lunar rock, core samples, pebbles, sand and dust from the Moon surface. It is estimated that Moon's crust formed 4.4by ago. A team of scientist have studied Apollo 14 zircon fragments. They put the age of Moon at 4.51by [12]. Matja Cuk et.al. finally has resolved this conundrum [2]. According to this research, from $3R_E$ to $45R_E$ , Moon does not have a smooth transit. Infact it is bumpy. It is chaotic, gets stuck in resonances and comes out of the resonances and gets stalled and resumes its tidal evolution. In fact Moon takes 3.267Gy to spirally expand from $3R_E$ to 45RE in fits and stalled manner. From $45R_E$ to $60.336R_E$, Moon smoothly coasts in 1.2Gy. This accelerated spiral expansion in the on-going phase results in present day velocity of recession of 3.7cm/y. As we see in abstract Sharma 2024, this consistency with LLR results resolves a long standing problem of mismatch between observed LOD curve and theoretical LOD curve. In this new E-M model, a precise match is obtained between the theory and observation.

## 6. Conclusion

The series of papers in CELMEC VII and these two abstracts ID 20609 and ID 20935 in COSPAR-2018 have set the stage for Advanced KM to be established as a well tested tool for further applications in Space Dynamics. I also envisage the application of this model in earth-quake predictions.


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