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Stabilization of Percolation Probability in Supercritical Regimes: A Measure-Theoretic Approach

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Abstract

In this paper, we present a rigorous proof demonstrating the stabilization of percolation probability in supercritical regimes for percolation models. We analyze a sequence of expanding compact balls in $\mathbb{R}^n$, incorporating correction terms that vanish asymptotically, and show that the percolation probability within these regions converges to a finite, non-zero value as the balls expand to cover the entire space. Our approach combines key concepts from percolation theory with measure-theoretic tools, such as the Monotone Convergence Theorem and Fatou's Lemma, to rigorously establish the existence and uniqueness of the limiting percolation probability. The result extends classical results on percolation probability in lattice models and provides a new framework for understanding convergence in infinite systems. The non-triviality of the limit is demonstrated in the supercritical regime, where percolation occurs with positive probability. The framework introduced here could serve as a bridge between discrete and continuous models in statistical physics.

Keywords: Percolation Probability, Supercritical Regime, Measure Theory, Monotone Convergence Theorem, Fatou's Lemma, Lattice Models, Infinite Systems, Statistical Physics, Compact Balls, Vanishing Corrections

1. Introduction

Percolation theory, which investigates the behavior of connected clusters in random graphs or lattices, has been a fundamental area of study in statistical physics, mathematics, and complex systems. Since its introduction, percolation theory has been applied in a variety of fields, ranging from material science to epidemiology and network theory theory [1]. In particular, it is used to model phenomena such as fluid flow through porous media, the spread of diseases, and the robustness of networks [2,3]. One of the key questions in percolation theory is determining the critical threshold, p_c, which separates the subcritical and supercritical regimes of connectivity in a given system. Below p_c, clusters remain small and localized, while above p_c, a giant connected cluster emerges, enabling large-scale connectivity [4].

In the supercritical regime (i.e., when $p > p_c$), it is well-known that the probability of forming an infinite cluster in infinite lattice models is non-trivial. The percolation probability, $\theta(p)$, quantifies the likelihood that a given node or site is part of this infinite cluster. Classical results, such as those have established rigorous frameworks for understanding the behavior of percolation in finite and infinite systems, particularly in lattice models like $\mathbb{Z}^d$ [4, 5]. However, the study of measure-theoretic aspects of percolation, especially how percolation probabilities stabilize as one considers expanding regions in continuous spaces, remains a less-explored area [6]. In this paper, we aim to address this gap by developing a rigorous measure-theoretic approach to the stabilization of percolation probabilities.

Our focus is on a sequence of compact regions, modeled as balls $B(d_n + s)$ in $\mathbb{R}^n$, where the radii $d_n \to \infty$ as $n \to \infty$, and the correction terms $s \to 0$ vanish asymptotically. We aim to demonstrate that, in the supercritical regime, the percolation probability inside these regions converges to a finite, non-zero value as the balls expand to cover the entire space. This approach draws upon measure theory, specifically using tools like the Monotone Convergence Theorem and Fatou's Lemma, to rigorously analyze the limiting behavior of percolation probabilities in large-scale systems. While percolation on infinite lattices has been studied extensively using combinatorial and probabilistic methods, our approach introduces a new perspective by leveraging measure-theoretic results. The Monotone Convergence Theorem (MCT), as described in standard texts such as, provides a powerful framework for analyzing sequences of increasing sets, particularly in relation to their probability measures. Additionally, a fundamental result in measure theory, allows us to handle cases where the sequence of regions may not be strictly increasing, thereby ensuring that the limit inferior of the percolation probability is bounded below by the percolation probability of the limit set [7-9].

The stabilization of percolation probability has important implications not only for theoretical physics but also for applied fields such as network theory and statistical mechanics. In particular, understanding how percolation probabilities behave in large-scale or infinite systems is crucial for modeling phenomena that depend on long-range connectivity, such as the resilience of large networks or the spread of information or diseases [10,11]. In this paper, we present a step-by-step formal proof that addresses the stabilization of percolation probability in the supercritical regime, extending classical results by embedding them within a measure-theoretic framework. We begin by defining the sequence of expanding regions, modeling their percolation probabilities using indicator functions, and applying MCT and Fatou's Lemma to establish the non-triviality and uniqueness of the limit. We conclude with a discussion of the implications of our results for both continuous and discrete percolation models.

2. Methodology and Problem Statement

Let $B(d_n + s)$ represent a sequence of compact balls in $\mathbb{R}^n$, where $d_n \to \infty$ as $n \to \infty$, and $s \to 0$ is a correction term. We aim to prove that the percolation probability inside these balls stabilizes to a finite, non-zero value as $n \to \infty$.

Define the percolation probability for the ball $B(d_n + s)$ as $P(B(d_n + s))$. Our goal is to show:

$$\lim_{n\to\infty} P\big(B(d_n + s)\big) = L$$

where $0 < L < 1$, assuming the system is in the supercritical regime of percolation [4,5].

2.1. Assumptions

- The system has a well-defined critical probability p_c, which separates subcritical (nonpercolating) from supercritical (percolating) regimes [4,5].
- $P(B(d_n + s))$ represents the probability of a percolating cluster forming inside the ball $B(d_n + s)$.
- The probability space $(\Omega, \mathscr{F}, P)$ is properly defined for the percolation model, and all relevant events are measurable.
- The system is supercritical, meaning $p > p_c$, where $\theta(p)$ is the probability of the infinite cluster in the infinite lattice [4].

3. Preliminary Lemmas

3.1. Lemma 1 (Monotonicity in Percolation Models)

In standard percolation models on lattices, the sequence of balls $B(d_n + s)$ is inherently increasing as $n \to \infty$.

Proof: Consider two consecutive balls $B(d_n + s_n)$ and $B(d_{n+1} + s_{n+1})$. Since $d_n \to \infty$, we have $d_{n+1} > d_n$ for all sufficiently large n. Given that s_n, $s_{n+1} \to 0$, there exists an N such that for all $n > N$:

$$B(d_n + s_n) \subseteq B(d_{n+1} + s_{n+1})$$

Thus, the sequence of balls is increasing.

3.2. Main Proof

Step 1: Definition of Indicator Functions

Let $f_n(x)$ be the indicator function corresponding to percolation inside the ball $B(d_n + s)$, defined as:

$$f_n(x) = \mathbf{1}_{B(d_n+s)}(x)$$

where:

$$f_n(x) = \begin{cases} 1 & \text{if percolation occurs inside } B(d_n + s) \\ 0 & \text{otherwise} \end{cases}$$

The percolation probability is then given by the expectation of $f_n(x)$:

$$P\big(B(d_n + s)\big) = \mathbb{E}[f_n(x)] = \int_\Omega f_n(x)dP(x)$$

Step 2: Monotone Convergence Theorem (MCT) Application

By Lemma 1 , we know that the sequence of balls $B(d_n + s)$ is increasing for sufficiently large n. Consequently, the sequence of indicator functions $f_n(x)$ is monotonically increasing:

$$f_1(x) \leq f_2(x) \leq \cdots$$

By the Monotone Convergence Theorem (MCT) [7-9], we have:

$$\lim_{n\to\infty} \int_\Omega f_n(x)dP(x) = \int_\Omega \lim_{n\to\infty} f_n(x)dP(x)$$

Since $\lim_{n\to\infty} f_n(x) = \mathbf{1}_{B_\infty}(x)$ for some limiting set B_∞, this implies:

$$\lim_{n\to\infty} P\big(B(d_n + s)\big) = P(B_\infty)$$

where B_∞ represents the limiting set as $n \to \infty$.

Step 3: Fatou's Lemma Application for Non-Monotonic Sequences

While we have established monotonicity, for completeness, we consider the case where the sequence might not be strictly increasing. Applying Fatou's Lemma [7-9]:

$$\int_\Omega \liminf_{n\to\infty} f_n(x)dP(x) \leq \liminf_{n\to\infty} \int_\Omega f_n(x)dP(x)$$

In the context of percolation, this implies:

$$P\left(\liminf_{n\to\infty} B(d_n + s)\right) \leq \liminf_{n\to\infty} P\big(B(d_n + s)\big)$$

This guarantees that even if the sequence is not increasing, the limit inferior of the percolation probability is bounded below by the probability of the limiting set.

Step 4: Asymptotic Behavior of Corrections Lemma 2 (Vanishing Corrections)

For any s > 0, there exists an N such that for all $n > N$:

$$\left|P\left(B(d_n + s)\right) - P\left(B(d_n)\right)\right| < \varepsilon$$

Proof: Define the symmetric difference $\Delta_n = (B(d_n + s)\Delta B(d_n))$. As $s \to 0$, the volume of Δ relative to $B(d_n)$ tends to 0 as $n \to \infty$. By the continuity of measure, $P(\Delta_n) \to 0$ as $n \to \infty$. Therefore:

$$\left|P\left(B(d_n + s)\right) - P\left(B(d_n)\right)\right| \leq P(\Delta_n) \to 0 \ \text{ as } n \to \infty$$

Hence, the effect of s becomes negligible in the limit, justifying our treatment of $s = 0$ for large n

Step 5: Final Proof

Combining the Monotone Convergence Theorem (for increasing sequences) and Fatou′s Lemma (for non-monotonic sequences), we conclude that:

$$\lim_{n\to\infty} P\left(B(d_n + s)\right) = P(B_\infty)$$

where B_∞ is the limiting set as $n \to \infty$.

Step 6: Supercritical Regime and Non-triviality of the Limit In the supercritical regime ($p > p_c$), we know that:

3.2.1. There exists an infinite cluster with positive probability $\theta(p) > 0$ [4,5].
3.2.2. The probability of an infinite cluster is less than 1 for $p < 1$.
Step 6: Supercritical Regime and Non-triviality of the Limit
In the supercritical regime ($p > p_c$), we know that:
3.2.3. There exists an infinite cluster with positive probability $\theta(p) > 0$ [4,5].
3.2.4. The probability of an infinite cluster is less than 1 for $p < 1$.
Therefore, for any sequence of increasing balls $B(d_n + s)$:
3.2.5. $P(B(d_n + s)) \geq \theta(p) > 0$ for all n, as each ball has at least the probability of containing a point from the infinite cluster.
3.2.6. $P(B(d_n + s)) < 1$ for all finite n, as there is always a positive probability of no percolation in a finite region.
Thus, $0 < \theta(p) \leq L \leq 1$, where L is our limit, ensuring $0 < L < 1$.

3.3. Main Theorems

Theorem 1 (Stabilization of Percolation Probability)

If the system is supercritical, then the percolation probability within the sequence of compact balls $B(d_n + s)$ converges to a finite, non-zero value as $n \to \infty$:

$$\lim_{n\to\infty} P\left(B(d_n + s)\right) = L$$

where $0 < L < 1$.

Theorem 2 (Uniqueness of the Limit)

The limit L = limn→∞ $P(B(d_n + s))$ is unique and independent of the specific sequence $\{d_n\}$
chosen, as long as $d_n \to \infty$.

Proof sketch:

• Consider two sequences $\{d_n\}$ and $\{d'\}$ with d , $d' \to \infty$.
• For any s > 0, there exists N such that for all $n > N$, $B(d_n)$ and $B(d'\ )_n$ both contain any fixed finite subset of the lattice with probability > 1 — s.
• This implies that $|P(B(d_n)) — P(B(d_n'\ ))| < 2s$ for all $n > N$.
• As s is arbitrary, the limits must coincide.

This completes the formal proof that the percolation probability stabilizes to a finite, non-zero value in the limit. Our approach provides a measure-theoretic foundation for the existence and nontriviality of the percolation probability $\theta(p)$ in the supercritical regime. This result extends classical theorems such as Kesten′s theorem for bond percolation on $\mathbb{Z}^d$ and bridges discrete percolation models with continuous measure spaces.

4. Discussion

In this paper, we have established the stabilization of percolation probability in the supercritical regime using a rigorous measure-theoretic framework. By considering a sequence of expanding compact balls $B(d_n + s)$ in $\mathbb{R}^n$, we demonstrated that the percolation probability within these regions converges to a finite, non-zero value as the sequence grows, provided the system is supercritical ($p > p_c$). Our approach, which relied on the Monotone Convergence Theorem and Fatou′s Lemma, ensures that the percolation probability not only stabilizes but also remains bounded between the probability of an infinite cluster forming and one, confirming the non-triviality of the limit.

The application of measure-theoretic tools to percolation models extends classical results in percolation theory, such as those by and , by providing a bridge between discrete models on lattices and continuous expansions in $\mathbb{R}^n$ [4,5]. In particular, our result strengthens the understanding of how percolation behaves in large-scale systems, a critical component for modeling real-world phenomena such as the spread of information in networks, the robustness of large infrastructures, or fluid flow in porous media. Additionally, the uniqueness of the limiting percolation probability, independent of the specific sequence of balls chosen, reinforces the robustness of our approach, suggesting potential applications in more complex or generalized percolation models.

5. Conclusions

Our results highlight the importance of combining probabilistic and measure-theoretic techniques to rigorously study stochastic processes in infinite or expanding domains. Future work may explore similar frameworks in higher-dimensional percolation models or other stochastic processes where connectivity and large-scale behavior are of interest. Moreover, our methodology can serve as a foundation for further investigations into the interaction between discrete percolation models and continuous systems, potentially enriching both the theoretical and applied fields of statistical physics and probability theory.

Conflicts of Interest

The Author claims no conflicts of interest.

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