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Research Article

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Solving Apollonius' Problem Through Relativity

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Abstract

We solve the problem of Apollonius by applying the Lorentz transformation within the framework of special relativity to go to a frame of simultaneity, where the solutions are the circumcenter. We transform back to get the solutions in a general frame. We show that the solutions in a general frame are the foci of the ellipse going through the centers of the given 3 circles.

Keywords: Apollonius' Problem, Conics, Relativity, Foci, Lorentz Transformation, Frame of Simultaneity

1. Apollonius' Problem and Lorentz Invariance

Apollonius problem can be stated as follows. Given 3 circles, find the circles that touch these three circles. Let C_A, C_B, C_C be the three given circles, with centers $A = [a_1, a_2]$, $B = [b_1, b_2]$, $C = [c_1, c_2]$ and with corresponding radii $\pm r_A = a_3, \pm r_B = b_3, \pm r_c = c_3$. We can view the problem in 3 dimensions, in particular in (2,1)-dim spacetime. There, the circles are given by the position vectors $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\vec{c} = (c_1, c_2, c_3)$. In (2,1)-dim the problem we need to solve is, where do the following 3 light cones meet

$$\begin{aligned}(x-a_1)^2+(y-a_2)^2&=(t-a_3)^2\\(x-b_1)^2+(y-b_2)^2&=(t-b_3)^2\\(x-c_1)^2+(y-c_2)^2&=(t-c_3)^2\end{aligned}\tag{1}$$

When we subtract these equations in pairs we obtain

$$\begin{aligned}x(b_1-a_1)+y(b_2-a_2)-t(b_3-a_3)&=\frac{b_1^2+b_2^2-b_3^2}{2}-\frac{a_1^2+a_2^2+-a_3^2}{2}\\x(c_1-b_1)+y(c_2-b_2)-t(c_3-b_3)&=\frac{c_1^2+c_2^2-c_3^2}{2}-\frac{b_1^2+b_2^2-b_3^2}{2}\\x(a_1-c_1)+y(a_2-c_2)-t(a_3-c_3)&=\frac{a_1^2+a_2^2-a_3^2}{2}-\frac{c_1^2+c_2^2-c_3^2}{2}\end{aligned}\tag{2}$$

Observe that this linear set of equations (2) is Lorentz invariant in the sense that it is formulated in terms of a relativistic inner product

$$\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2-u_3v_3\tag{3}$$

The same is true for the set of equations that defines the problem, Eq.1. With this observation, the solution to Apollonius' problem can be obtained in the following manner. Solve for the plane through the points and intersect it with the line given by the linear set of equations, Eq.2. To get the solutions, we only need to compute the relativistic length from the point of intersection with any point of the given points, using the relativistic inner product, Eq.3. We need to move in the direction of the line given by Eq.2 up and down, starting from the point of intersection, to get the solutions. Let's explain why this is true.

Suppose the circles are of the same size for example, then the solution is the circumcenter.

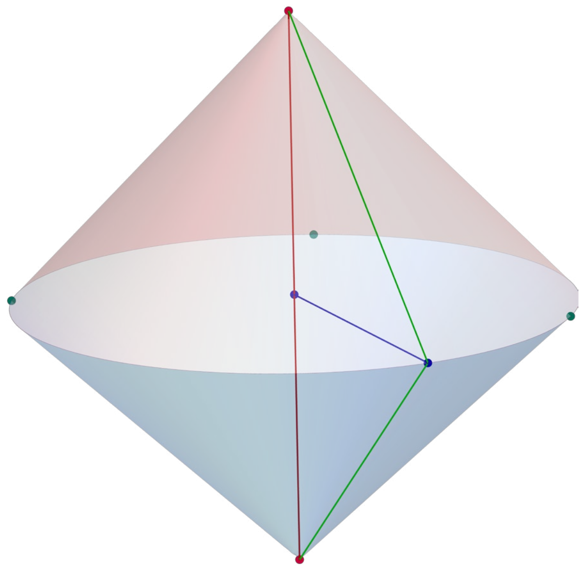

Figure 1: The 3 given circles are being displayed as 3 green points in (2,1)-dim. There is a point in time where the three given circles are shrunk to a point and since the circles are the same size, these points are on a horizontal plane. The circumcenter of the circle going through the given green points is shown in blue. The red points are the solutions in (2,1)-dim spacetime and have, in the case that the green circles are the same size, the same x, y coordinates as the blue circumcenter.

In 3 dim the line goes up and down from the circumcenter by a length $\sqrt{-x^2 - y^2 + z^2}$ where here $-x^2 - y^2 = 0$.

In the general case, where the circles are not the same size,

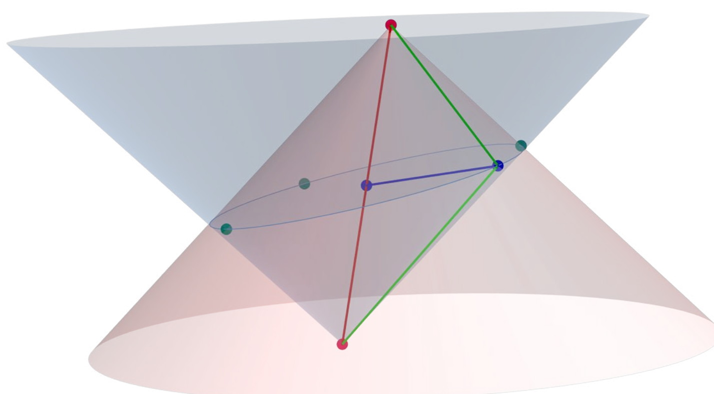

Figure 2: The 3 given circles are shown as 3 green points. Since they are of different sizes they are not an a horizontal plane. The red points are the solutions to the problem. They are the circles that touch the 3 given circles. Also, the associated cones of the red points are being displayed. They meet in an ellips, which goes through the 3 green points.

we can just Lorentz transform the problem into a situation where the circles are the same size. This is the frame of simultaneity. We can do this, since the linear set of equations (2) is Lorentz invariant.

We can solve of the (2,1)-circumcenter through intersecting the line given by Eq.2 and the plane through the 3 given points. This is true because, if in one frame, the frame of simultaneity, the intersection is the circumcenter, then in any Lorentz transformed frame the circumcenter becomes just the (2,1)-circumcenter. We can view this as the definition of the (2,1)-circumcenter. The solutions are just the relativistic length up and down analogues to the case when the line is vertical in the case the given circles are all the same size.

2. Foci of the Ellipse

The solutions to Apollonius problem are the foci of the ellipse through the given 3 points.

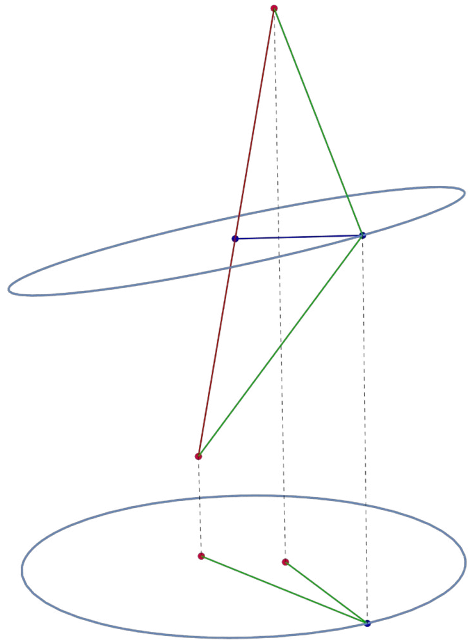

Figure 3: The (2,1)-dim configuration is being projected on a horizontal plane.

The ellipse is just the Lorentz transformed circle going through the centers of the 3 given circles. We show in 2-dim that the ellipse goes through the centers of the three given circles and that the centers of the solution circles are the foci of the ellipse.

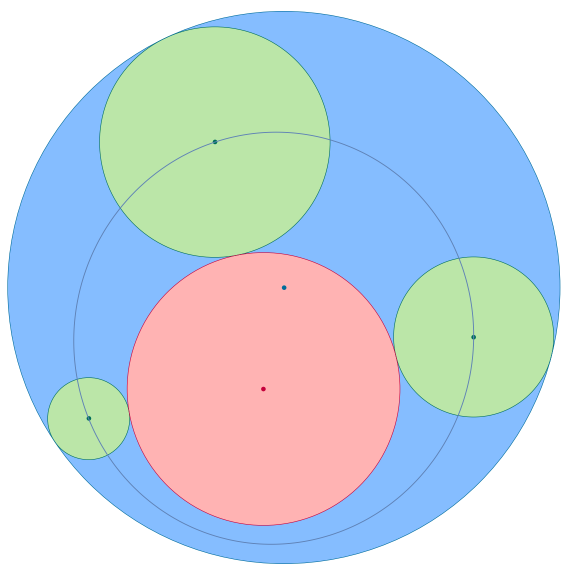

Figure 4: The blue and the red circle are the solutions to the problem, which touch the 3 given green circles. The ellipse goes through the centers of the 3 given circles and the centers of the solutions are the foci of the ellipse.

3. Two Given Circles Inside Another Given Circle

The problem of Apollonius can be stated in many different situations and these can limit the number of solutions to the point there aren't any. The given circles can be inside each other for instance. We consider the scenario in which two given circles are inside the third given circle

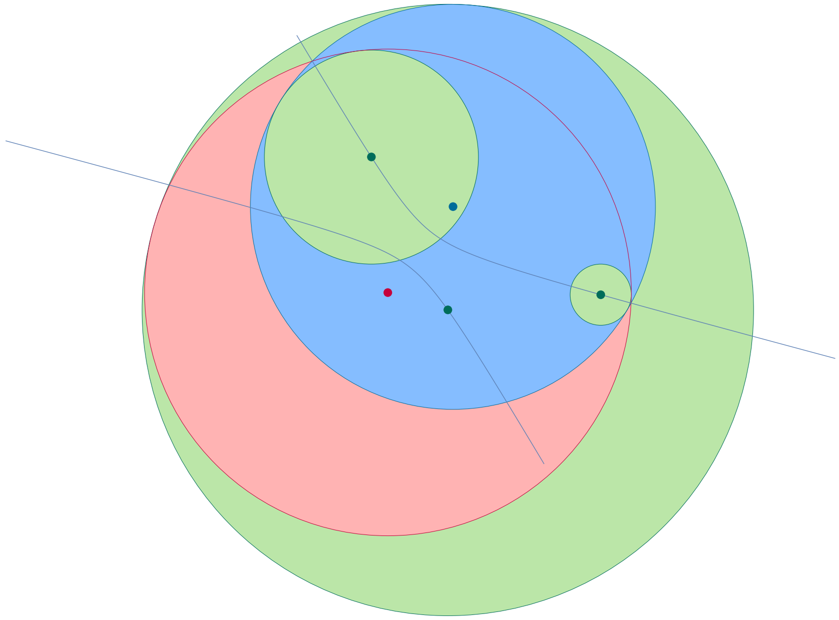

Figure 5: The 3 given circles are shown in green and the solution circles in red and blue. When 2 of the three given circles are inside the third given green circle, the conic section that goes through the centers of the given 3 circles is an hyperbola. The centers of the solution circles are the foci of the hyperbola.

In this case the conic section is a hyperbola instead of an ellipse. The hyperbola goes through the centers of the given 3 circles, and the centers of the blue and the red circle are the foci of the hyperbola. In general, there are maximally 8 solutions, so 4 pairs, since the radii of the given circles correspond to 2 points in (2,1)-dim spacetime.

4. The Lorentz Transformation

We can construct the Lorentz transformation in (2,1)-dim by combining three (2,1)-orthonormal basis vectors $e_1 = \frac{u}{\sigma(u)}, e_2 = \frac{u\times(u\times v)}{\sigma(u\times(u\times v))}$, $e_3 = \frac{u\times v}{\tau(u\times v)}$ into a matrix Λ, such that the basis vectors form the

$$\Lambda = e_1 e_2 e_3 = \begin{pmatrix} (e_1)_1 & (e_2)_1 & (e_3)_1 \\ (e_1)_2 & (e_2)_2 & (e_3)_2 \\ (e_1)_3 & (e_2)_3 & (e_3)_3 \end{pmatrix}$$

Using the above notation, the Lorentz transformation is then represented by the matrix

$$\Lambda = \frac{u}{\sigma(u)} \frac{u\times(u\times v)}{\sigma(u\times(u\times v))} \frac{u\times v}{\tau(u\times v)}$$

where $u = \overrightarrow{AB}$, and $v = \overrightarrow{AC}$, and $\sigma(u) = \sqrt{u_1^2 + u_2^2 - u_3^2}$ and $\tau(u) = \sqrt{-u_1^2 - u_2^2 + u_3^2}$, and where $v \times w = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, -(v_1w_2 - v_2w_1))$ is a (2,1)-dim cross product. Note that the (2,1)-dim cross product ensures that the (2,1)-dim cross product ensures that the basis vectors are (2,1)-dim perpendicular and the τ and σ functions normalise the vectors. Usually the Gram-Schmidt process is used for creating perpendicular vectors [32]. The obtained Lorentz transformation allows us to go to the frame of simultaneity, where the 3 given circles are the same size.

5. Conclusion

We have proposed a way of calculating the solutions of Apollonius' problem by solving two matrix equations and computing a relativistic length. The solutions are the foci of the corresponding conic going through the centers of the 3 given circles in 2-dim.

References

1. Coxeter, H. S. M. (1968). The problem of Apollonius. *The American Mathematical Monthly, 75*(1), 5-15.
2. Dörrie, H. (1965). The tangency problem of Apollonius. *100 Great Problems of Elementary Mathematics: Their History and Solutions,* 154-160.
3. Coolidge, J. L. (1916). *A Treatise on the Circle and the Sphere*. Clarendon Press.

4. Coxeter, H. S. M., Greitzer, S. L. (1967). Geometry Revisited. Washington: MAA.
5. Needham, T. (2023). *Visual complex analysis*. Oxford University Press.
6. Hofmann-Wellenhof, B., Legat, K., & Wieser, M. (2011). *Navigation: principles of positioning and guidance*. Springer Science & Business Media.
7. Schmidt, R. O. (1972). A new approach to geometry of range difference location. *IEEE Transactions on Aerospace and Electronic Systems*, (6), 821-835.
8. Coaklay, G. W. (1860). Analytical solutions of the ten problems in the tangencies of circles; and also of the fifteen problems in the tangencies of spheres. *The Mathematical Monthly, 2,* 116-126.
9. Court, N. A. (1961). Historically Speaking,—: The problem of Apollonius. *The Mathematics Teacher, 54*(6), 444-452.
10. Gabriel-Marie, F. (1912). Exercices de gomtrie, comprenant l'expos des mthodes gomtriques et 2000 questions rsolues (in French). Tours: Maison A. Mame et Fils, 18-20.
11. van Roomen, A. *Problema Apolloniacum Quo Datis Tribus Circulis, Quaeritur Quartus Eos Contingens: Antea Ab Illustri Viro D. Francisco Vieta... ad construendum propositum*. Fleischmann.
12. Newton, I. (1974). DT Whiteside (ed.). The Mathematical Papers of Isaac Newton, Cambridge: Cambridge University Press, 164.
13. Newton, I., & Whiteside, D. T. (2008). The mathematical papers of Isaac Newton. *The Mathematical Papers of Isaac Newton.*
14. Bold, B. (1982). *Famous problems of geometry and how to solve them*. Courier Corporation.
15. Boyer, C. B., & Merzbach, U. C. (2011). *A history of mathematics*. John Wiley & Sons.
16. Pedoe, D. (1970). The Missing Seventh Circle. *Elemente der Mathematik, 25*, 14-15.
17. Bruen, A., Fisher, J. C., & Wilker, J. B. (1983). Apollonius by inversion. *Mathematics Magazine, 56*(2), 97-103.
18. Zlobec, B. J., & Kosta, N. M. (2001). Configurations of cycles and the Apollonius problem. *The Rocky Mountain Journal of Mathematics*, 725-744.
19. Euler, L. (1790). Solutio facilis problematis, quo quaeritur circulus, qui datos tres circulos tangat. *Nova Acta Academiae Scientiarum Imperialis Petropolitanae,* 95-101.
20. Gauss, C. F. (1873). Werke, 4. Band (in German) (reprinted in 1973 by Georg Olms Verlag (Hildesheim) ed.). Gttingen: Kniglichen Gesellschaft der Wissenschaften, 399-400.
21. Knight, R. D. (2005). The Apollonius contact problem and Lie contact geometry. *Journal of Geometry, 83*, 137-152.
22. Ogilvy, C. S. (1990). *Excursions in geometry.* Courier Corporation.
23. Eisenbud, D., & Harris, J. (2016). *3264 and all that: A second course in algebraic geometry*. Cambridge University Press.
24. Stevanovic, M. R., Petrovic, P. B., & Stevanovic, M. M. (2017). Radii of Circles in Apollonius' Problem. In *Forum Geometricorum* (Vol. 17, pp. 359-372).
25. Drechsler, K., & Sterz, U. (1999). APOLLONIUS'CONTACT PROBLEM IN n–SPACE IN VIEW OF ENUMERATIVE GEOMETRY. *Acta Math. Univ. Comenianae, 68*(1), 37-47.
26. Fitz-Gerald, J. M. (1974). A note on a problem of Apollonius. *Journal of Geometry, 5*, 15-26.
27. Eppstein, D. (2001). Tangent spheres and triangle centers. *The American Mathematical Monthly, 108*(1), 63-66.
28. Pedoe, D. (1967). On a theorem in geometry. *The American Mathematical Monthly, 74*(6), 627-640.
29. Mumford, D., Series, C., Wright, D. J., & Wright, D. (2002). Indra's pearls: The vision of Felix Klein. Cambridge University Press.
30. Spiesberger, J. L. (2004). Geometry of locating sounds from differences in travel time: Isodiachrons. *the Journal of the Acoustical Society of America, 116*(5), 3168-3177.
31. Lewis, R. H., & Bridgett, S. (2003). Conic tangency equations and Apollonius problems in biochemistry and pharmacology. *Mathematics and Computers in Simulation, 61*(2), 101-114.
32. Cheney, W., & Kincaid, D. (2009). Linear algebra: Theory and applications. *The Australian Mathematical Society, 110*, 544-550.
33. Vrahatis, M. N. (2024). Generalization of the Apollonius theorem for simplices and related problems. *arXiv preprint arXiv:2401.03232*.
34. De Loera, J., Goaoc, X., Meunier, F., & Mustafa, N. (2019). The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg. *Bulletin of the American Mathematical Society, 56*(3), 415-511.
35. Har-Peled, S., & Robson, E. W. (2023). On the Width of the Regular $ n $-Simplex. *arXiv preprint arXiv:2301.02616.*
36. Vrahatis, M. N. (2024). Towards the mathematical foundation of the minimum enclosing ball and related problems. *arXiv preprint arXiv:2402.06629.*
37. Nocar, D., & Dofková, R. (2020). APOLLONIUS'PROBLEMS IN SECONDARY EDUCATION USING ICT. In *EDULEARN20 Proceedings* (pp. 3572-3580). IATED.