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# Results of a Korteweg-de Vries Equation Generated by a Semigroup of Linear Operators

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**Abstract**
*In this study, partial contraction mapping with ω-order preservation (ω-OCPn,) is shown to provide a broad class of semilinear initial value issues. By starting with certain conclusions pertaining to such fractional powers, we investigated the application of fractional powers of unbounded linear opera- tors. The fractional powers of A for $0 < \alpha \leq 1$ are defined on the assumption that A is the infinitesimal generator of an analytic semigroup in a Banach space X, $0 \in \rho(A)$. We demonstrated that the closed linear operator Aα with domain $D(A^\alpha) \supset D(A)$ is dense in X. Finally, we determined that the operator is Holder continuous, continuous, and bounded.*

**Keywords:** ω-OCP$_n$, Strongly Elliptic, C0-semigroup, Analytic Semigroup

AMS (MOS) Subject Classifications: 15A60, 65F35, 65J05.

## 1. Introduction

Consider the Korteweg-de Vries equation

$$\begin{cases} u_1 + u_{xxx} + uu_x = 0 \quad t \geq 0 \quad -\infty < x < \infty \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

such that all function are real valued. For every real s we introduce a Hilbert space $H^s(\mathbb{R})$ as follows: Let $u \in L^2(\mathbb{R})$ and set

$$\|u\|_s = \left(\int (1+\xi^2)^s |\hat{u}(\xi)^2 d\xi\right)^{1/2} \tag{1.2}$$

The linear space of functions $u \in L^2(\mathbb{R})$ for which $||u||_s$ is nite is a pre-Hilbert space with the scalar product

$$(u,v) = \int (1+\xi^2)^s \hat{u}(\xi)\bar{\hat{u}}(\xi) d\xi. \tag{1.3}$$

The completion of this space with respect to norm $\| \|_s$ is a Hilbert space which is denoted by $H^s(\mathbb{R})$. It is clear that $H^0(\mathbb{R}) = L^2(\mathbb{R})$. The scalar product and norm in $L^2(\mathbb{R})$ is denoted by (,) and $\| \|_0$. Furthermore, it is easy to check that the spaces $H^s(\mathbb{R})$ with $s = n$ coincide with the spaces $H^n(\mathbb{R})$, $n \geq 1$. Suppose $B_r$ is the ball of radius $r > 0$ in $Y$ centered at the origin and consider the family of operators $A(v)$, $v \in B_r$. Because of the special form of the family $A(v)$, $v \in B_r$, it follows that it suffices to state the following three conditions:

($P_1$) The family $A(v)$, $v \in B_r$, is a stable family in $X$.

(P2) There is an isomorphism of $Y$ onto $X$ such that for every $v \in B_r$ $SA(v)S^{-1} - A(v)$ is a bounded operator in $X$ and

$$\|SA(v)S^{-1} - A(v)\| \leq C_1 \; for \; all \; v \in B_r. \tag{1.4}$$

($P_3$) For each $v \in B_r$, $D(A(v)) \supset Y$, $A(v)$ is a bounded linear operator from $Y$ into $X$ and

$$\|A(v_1) - A(v_2)\|_{Y \to X} \leq C_2 \|v_1 - v_2\|. \tag{1.5}$$

Furthermore, if $\|u_0\|_s < r$ and $v \in B_r$, then

$$\begin{aligned} \|A(v)u_0\| &\leq \|D^3 u_0\| + \|vDu_0\| \\ &\leq \|D^3 u_0\| + \|v\|_\infty \|Du_0\| \\ &\leq \|u_0\|_3(1+r) \leq r(1+r) = k. \end{aligned} \tag{1.6}$$

Suppose $X$ is a Banach space, $X_n \subseteq X$ is a finite set, $\omega - OCP_n$ the $\omega$-order preserving partial contraction mapping, $M_m$ be a matrix, $L(X)$ be a bounded linear operator on $X$, $P_n$ a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of $A$. This paper consist of results of $\omega$- order preserving partial contraction mapping generating a Korteweg-de Vries equation. In and Akinyele *et al.* obtained differentiable and analytical conclusions on $\omega$-order preserving partial contraction mapping in semigroup of linear operator [1,2]. They also described $\omega$-order reversing partial contraction mapping as a compact semigroup of linear operator. An operator calculus for infinitesimal semigroup generators was presented by Balakrishnan [3]. Ba-nach created and first proposed the idea of Banach spaces [4]. The nonlinear Schrodinger evolution equation was created by Brezis and Gallouet [5]. A resolvent method to the stability operator semigroup was presented by Chill and Tomilov [6]. Davies discovered the spectrum of linear operators [7]. For equations of linear evolution, Engel and Nagel presented the one-parameter semigroup in their paper [8]. As well as introducing dual properties of $\omega$-order reversing partial contraction mapping in semigroup of linear operator in Omosowon et al. produced some analytical results of semigroup of linear operator with dynamic boundary conditions [9,10]. Pazy reported asymptotic behavior of an abstract evolution's solution and various applications, he obtained a class of evolution's semi-linear equations [11,12]. Rauf and Akinyele created $\omega$-order preserving partial contraction mapping and acquired its qualities [13]. Also in Rauf et al. established some results of stability and spectra properties on semigroup of linear operator [14]. Vrabie demonstrated a few applications of the C0-semigroup's findings [15]. Yosida derived several conclusions on the differentiability and representation of a linear operator one-parameter semigroup [16].

## 2. Preliminaries

**Definition 2.1** ($C_0$-Semigroup) [15]
A $C_0$-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

**Definition 2.2** ($\omega$-$OCP_n$) [13]
A transformation $\alpha \in P_n$ is called $\omega$-order preserving partial contraction mapping if $\forall x, y \in$ Dom$\alpha$: $x \leq y \Longrightarrow \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t + s) = T(t)\, T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

**Definition 2.3** (Evolution Equation) [12]
An evolution equation is an equation that can be interpreted as the differ- ential law of the development (evolution) in time of a system. The class of evolution equations includes, first of all, ordinary differential equations and systems of the form

$u = f(t, u)$, $u = f(t, u, u)$,

etc., in the case where u(t) can be regarded naturally as the solution of the Cauchy problem; these equations describe the evolution of systems with finitely many degrees of freedom.

**Definition 2.4** ( Mild Solution) [11]
A continuous solution $u$ of the integral equation.

$$u(t) = T(t - t_0)u_0 + \int_{t_0}^{t} T(t - s) f(s, u(s)) ds$$

will be called a mild solution of the initial value problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t, u(t)),\ t > t_0 \\ u(t_0) = u_0 \end{cases}$$

if the solution is a Lipschitz continuous function.

**Definition 2.5** (Analytic Semigroup) [15]
We say that a $C_0$-semigroup $\{T(t); t \geq 0\}$ is analytic if there exists $0 < \theta \leq \pi$, and a mapping $S : \overline{\mathbb{C}}_\theta \to L(X)$ such that:
(i) $T(t) = S(t)$ for each $t \geq 0$;
(ii) $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \overline{\mathbb{C}}_\theta$;
(iii) $lim_{z1 \in \overline{\mathbb{C}}_\theta, z1 \to 0} S(z_1)x = x$ for $x \in X$; and
(iv) the mapping $z_1 \to S(z_1)$ is analytic from $\overline{\mathbb{C}}_\theta$ to $L(X)$. In addition, for each $0 < \delta < \theta$, the mapping $z_1 \to S(z_1)$ is bounded from $\mathbb{C}_\delta$ to $L(X)$, then the $C_0$-Semigroup $\{T(t); t \geq 0\}$ is called analytic and uniformly bounded.

**Definition 2.6** (Strongly Elliptic) [1]
The operator $A(x, D)$ is strongly elliptic if there exists a constant $C > 0$ such that

$$Re(-1)^m A^1(x, \xi) \geq C|\xi|^{2m}$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$.

**Example 1**
For every 2 × 2 matrix in $[M_m(\mathbb{R}^n)]$.
Suppose

$$A = \begin{pmatrix} 2 & 0 \\ \Delta & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then we have

$$e^{tA} = \begin{pmatrix} e^{2t} & I \\ e^{\Delta t} & e^{2t} \end{pmatrix}.$$

**Example 2**
For every 3 × 3 matrix in $[M_m(\mathbb{C})]$, we have for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on $X$.

Suppose we have

$$A = \begin{pmatrix} 2 & 2 & I \\ 2 & 2 & 2 \\ \Delta & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA\lambda}$, then we have

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & I \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{\Delta t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

**Example 3**

Let $X = C_{ub}(\mathbb{N} \cup \{0\})$ be the space of all bounded and uniformly continuous function from $\mathbb{N} \cup \{0\}$ to $\mathbb{R}$, endowed with the sup-norm $\| \bullet \|_\infty$ and let $\{T(t);\, t \in \mathbb{R}_+\} \subseteq L(X)$ be defined by

$$[T(t)f](s) = f(t+s)$$

For each $f \in X$ and each $t, s \in \mathbb{R}_+$, one may easily verify that $\{T(t);\, t \in \mathbb{R}_+\}$ satisfies Examples 1 and 2 above.

**Lemma 2.1**

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of class $C^m$ and let $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$ where $1 \le r, q \le \infty$. For any integer $j$, $0 \le j < m$ and any $j/m \le \vartheta \le 1$ we have

$$\|D^j u\|_{0,p} \le C\|u\|_{m,r}^{\vartheta}\|u\|_{0,q}^{1-\vartheta} \tag{2.1}$$

provided that

$$\frac{1}{p} = \frac{j}{n} + \vartheta\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\vartheta)\frac{1}{q} \tag{2.2}$$

and $m - j - \frac{n}{r}$ is not a nonnegative integer, the (2.1) holds with $\vartheta = \frac{j}{m}$.

## 3. Main Results

This section presents the semigroup of linear operator's results by creating a Korteweg-de Vries equation using $\omega$-$OCP_n$:

**Theorem 3.1**

Let $A : D(A) \subseteq H^s(\mathbb{R}) \to H^s(\mathbb{R})$ be the infinitesimal generator of a $C_0$- semigroup $\{T(t)_{t\ge 0}\}$ where $A \in \omega - OCP_n$. Then we have:

(i) For $t \ge s$, $H^s(\mathbb{R}) \supset H^t(\mathbb{R})$ and $\|u\|_t \ge \|u\|_s$ for $u \in H^t(\mathbb{R})$.

(ii) For $H^s(\mathbb{R}) \subset C(\mathbb{R})$ and for $u \in H^s(\mathbb{R})$,

$$\|u\|_\infty \le C\|u\|_s \tag{3.1}$$

where $\|u\|_\infty = \sup\{|u(x)| : x \in \mathbb{R}\}$.

**Proof**

Part (i) is obvious from the definitions and the elementary inequality

$$(1+\xi^2)' \ge (1+\xi^2)^s \ for\ t \ge s\ and\ \xi \in \mathbb{R}.$$

From the Cauchy-Schwarz inequality we have,

$$|u(x)| = \left|\frac{1}{\sqrt{2\pi}}\int e^{tx\xi}\hat{u}(\xi)d\xi\right| \le \frac{1}{\sqrt{2\pi}}\left(\int \frac{d\xi}{(1+\xi^2)}\right)^{1/2}\left(\int (1+\xi^2)^s|\hat{u}(\xi|^2 d\xi\right)^{1/2} = C\|u\|_s$$

Therefore, that the integral defining $u$ in terms of $\hat{u}$ converges uniformly and $u$ is continuous.

Moreover,

$$\|u\|_\infty \le C\|u\|_s.$$

Hence the proof is completed.

**Theorem 3.2**

Suppose $A : D(A) \subseteq X \to X$ is a real valued function such that $A \in \omega - OCP_n$. For every $v \in Y$ the operator $A(v) = A_0 + A_1(v)$ is the infinitesimal generator of a $C_0$-semigroup $T_v(t)$ on $X$ satisfying

$$\|T_v(t)\| \leq e^{\beta t} \tag{3.2}$$

for every $\beta \geq \beta_0 (v) = C_0\|v\|$, where $C_0$ is a constant independent of $v \in Y$.

**Proof**

we note first tat since $v \in H^s(\mathbb{R})$, $Dv \in H^{s-1}(\mathbb{R})$ and since $s \geq 3$, it follows from Theorem 3.1 that $Dv \in L^\infty(\mathbb{R})$ and that $\|Dv\|_\infty \leq C\|Dv\|_{s-1} \leq C\|v\|_s$.

Now, for every $u \in H^1(\mathbb{R})$ we have

$$\begin{aligned}(A_1(v)u, u) &= \int vDu \cdot udx = \frac{1}{2}\int vDu^2dx = \frac{1}{2}\int Dvu^2dx \\ &\geq -\frac{1}{2}\|Dv\|_\infty\|u\|^2 \geq -C_0\|v\|_s\|u\|^2.\end{aligned}$$

Therefore, $A_1(v) + \beta I$ is dissipative for all $\beta \geq \beta_0(v) = C_0\|v\|_s$. Since $A_0$ is skew-adjoint, $A_0 + A_1(v) + \beta I$ is also dissipative for $\beta \geq \beta_0(v)$. Moreover,

$$\|(A_1(v) + \beta I)u\| \leq \|vDu\| + \beta\|u\| \leq \|u\|_\infty\|Du\| + \beta\|u\|. \tag{3.3}$$

Using integration by parts, it is not difficult to show that for every $u \in H^3(\mathbb{R})$ we have $\|Du\| \leq \|u\|^{2/3}\|D^3u\|^{1/3}$ and by polarization we obtain for every $\varepsilon > 0$,

$$\|Du\| \leq \varepsilon\|D^3u\| + C(\varepsilon)\|u\|. \tag{3.4}$$

Choosing $\varepsilon = \frac{1}{2}\|v\|_\infty$ and substituting (3.4) into (3.3) yields

$$\|(A_1(v) + \beta I)u\| \leq \frac{1}{2}\|A_0u\| + C\|u\| \tag{3.5}$$

for all $u \in D(A_0)$ and $A \in \omega - OCP_n$.

Therefore, we have that $A_0 + A_1(v) + \beta I = A(v) + \beta I$ is the infinitesimal generator of a $C_0$-semigroup of contractions of $X$ for every $\beta \geq \beta_0(v)$.

Hence, $A(v)$ is the infinitesimal generator of a $C_0$-semigroup $T_v(t)$ and this achieved the proof.

**Theorem 3.3**

Assume $A : D(A) \subseteq X \to X$ is a real valued function such that $A \in \omega - OCP_n$. Let $f \in H^s(\mathbb{R})$, $s > 3$ and let $T = (\Delta^s M_f - M_f \Delta^3)\Delta^{1-s}$. Then
$T$ is a bounded operator on $X = L^2(\mathbb{R})$ and

$$\|T\| \leq C\|grad\ f\|_{s-1}. \tag{3.6}$$

**Proof**

The Fourier transform of $T$ is the integral operator with Kernel $K(\xi, \eta)$ given by

$$K(\xi, \eta) = \{(1 + \xi^2)^{s/2} - (1 + \eta^2)^{s/2}\}\hat{f}(\xi - \eta)(1 + \eta^2)^{(s-1)/2}$$

since

$$|(1 + \xi^2)^{s/2} - (1 + \eta^2)^{s/2}| \leq s|\xi - \eta|(1 + \xi^2)^{(s-1)/2} + (1 + \eta^2)^{(s-1)/2}$$

we have

$$K(\xi,\eta) \le s(1+\xi^2)^{(s-1)/2}|\xi-\eta|\hat{f}(\xi-\eta)(1+\eta^2)^{(1-s)/2}+s|\xi-\eta|\hat{f}(\xi-\eta) = k_1(\xi,\eta)+k_2(\xi,\eta).$$

To show that $T$ is bounded, it suffices to show that operators $T_1$ and $T_2$ with Kernels $k_1(\xi, \eta)$ and $k_2(\xi, \eta)$ are bounded. Using the inverse Fourier transform we find that

$$T_1 = s\Delta^{s-1}M_g\Delta^{1-s}, \quad T_2 = sM_g \tag{3.7}$$

Where $M_g$ is the multiplication operator by the function $g$ for which $\hat{g}(\xi) = |\xi|\hat{f}(\xi)$. From (ii) of Theorem 3.1, it follows that

$$\|g\|_\infty \le C\|g\|_{s-1} \le C\|grad\ f\|_{s-1}. \tag{3.8}$$

Now,

$$\|T_1u\| = s\|\Delta^{s-1}M_g\Delta^{1-s}u\| = s\|M_g\Delta^{1-s}u\|_{s-1} \le s\|g\|_\infty\|u\| \tag{3.9}$$

and

$$\|T_2u\| = s\|gu\| \le s\|g\|_\infty\|u\|. \tag{3.10}$$

Therefore both $T_1$ and $T_2$ are bounded operators in $X$. Combining (3.8) with (3.7) and (3.10) yields the desired estimate (3.6). Hence the proof is completed.

**Theorem 3.4**
Let $A$: $D(A) \subseteq X \to X$ be the infinitesimal generator of a $C_0$-semigroup $\{T_v(t)_{t\ge0}\}$. For every $r > 0$, the family of operators $A(v)$, $v \in B_r$ satisfies the conditions $(P_1) - (P_3)$.

**Proof**
Suppose $r > 0$ is fixed. From Theorem 3.2, it follows that if $\beta \ge C_0 r$, $A(v)$ is the infinitesimal generator of a $C_0$-semigroup $T_v(t)$ satisfying $\|T_v(t)\| \le e^{\beta t}$ and therefore $A(v)$, $v \in B_r$ is a stable family in $X$.

Assume $S = \Delta^s$ is an isomorphism of $Y = H^s(\mathbb{R})$ onto $X = L^2(\mathbb{R})$. A simple computation shows that for $u, v \in Y$ we have

$$\begin{aligned}(SA(v)S^{-1} - A(v))u &= (S(vD)S^{-1} - vD)u \\ &= (Sv - vS)S^{-1}Du\end{aligned}$$

and therefore by Theorem 3.3, we have

$$\begin{aligned}\|(SA(v)S^{-1} - A(v))u\| &= \|(\Delta^s M_v - M_v\Delta^s)\Delta^{t-s}\Delta^{-1}\Delta u\| \\ &\le \|(\Delta^s M_v - M_v\Delta^s)\Delta^{1-s}\|\|\Delta^{-1}\Delta u\| \\ &\le C\|grad\ v\|_{s-1}\|u\| \le C\|v\|_Y\|u\|.\end{aligned}$$

Since $Y$ is dense in $X$ it follows that $\|SA(v)S^{-1} - A(v)\| \le C\|v\|_r \le C_r$ and (P2) is satisfied. Finally, since $s \ge 3$, $D(A(v)) \supset Y$ for every $u \in Y$, $A \in \omega - OCP_n$ and $v \in B_r$, we have

$$\begin{aligned}\|A(v)u\| \le \|\Delta^3 u\| + \|v\Delta u\| &\le \|\Delta^3 u\| + \|v\|_\infty\|\Delta u\| \\ &\le (1 + C\|v\|_s)\|u\|_s \le (1 + Cr)\|u\|_r\end{aligned}$$

and therefore $A(v)$ is bounded operator from $Y$ into $X$. Moreover if $v_1, v_2 \in B_r$, $v \in Y$, then

$$\|(A(v_1) - A(v_2))u\| = \|(v_1 - v_2)\Delta u\|$$
$$\leq \|v_1 - v_2\|\|\Delta u\|_\infty \leq C\|v_1 - v_2\|\|u\|_Y$$

and the proof is competed.

## 4. Conclusion

It has been demonstrated in this study that various Korteweg-de Vries equations can be generated by partial contraction mapping with $\omega$-order preservation.


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