

## R-Algebras of Dimension 1, 2, 4 or 8

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### Abstract

We will show that it is possible to find a few parameters that describes several "interesting" R-Algebras of dimension 4 and 8 (we will recall the very easy and well known 1 and 2 cases).

### 1. Introduction

We will deal with finite dimensional hypercomplex numbers, that is to say,  $\mathbb{R}$ -algebras whose elements can be written (in dimension  $n$ ):

$$x = \sum_{i=0}^{n-1} a_i e_i$$

Where  $a_i \in \mathbb{R}$ , and  $e_i$  are elements of a basis of the underlying vector space in particular  $e_0 = 1$ ; by bilinearity of the multiplication, such an algebra is fully described by the multiplication table of a basis.

### 2. Dimension 1

This case is very simple: there exists only one  $\mathbb{R}$ -Algebra of dimension 1:  $\mathbb{R}$  itself, the natural basis is [1].

### 3. Dimension 2

The dimension 2 is very well known, a basis of such an algebra is [1,  $e$ ], and the multiplication table is given by :

$\cdot$	1	$e$
1	1	$e$
$e$	$e$	$e^2$

$\mathbb{R}^2$

Where  $e^2 = a + b \cdot e$  for some  $a$  and  $b$  two real numbers. This algebra depends only on the value of  $e^2$ , and it is well known that there exists only 3 different such algebras (up to isomorphism) according to the value of  $e^2 \in \{-1, 0, 1\}$ , when  $e = -1$ , it is usually noted  $i$ , when  $e = 0$ , it is usually noted  $\varepsilon$ , when  $e = 1$ , it is usually noted  $j$ :

$\cdot$	1	$i$
1	1	$i$
$i1$	$i$	-1

Complex Numbers :  $\mathbb{C}$

$\cdot$	1	$\varepsilon$
1	1	$\varepsilon$
$\varepsilon$	$\varepsilon$	0

Dual Numbers :  $\mathbb{D}_1$

$\cdot$	1	$j$
1	1	$j$
$j$	$j$	1

Perplex<sup>1</sup> Numbers :  $\mathbb{C}$

### 4. Dimension 4

Starting with dimension 4 there are too many possibilities to give a complete list of the  $\mathbb{R}$ -algebras, so we add some constraints<sup>2</sup>: Elements of a  $\mathbb{R}$ -algebra of dimension 4 can be written:  $z = a_0 + a_1 \cdot e_1 + a_2 \cdot e_2 + a_3 \cdot e_3$ , where  $a_i \in \mathbb{R}$ . For sake of simplicity we will not use the  $\cdot$  symbol anymore.

#### 4.1 Parametrization for Dimension 4

The following rules are very easy to encode in a spreadsheet, to experiment the different possibilities and to create new algebras

(particularly the algebras not mentioned here (a few), but beware of the existence of an isomorphism with well-known algebra.

- ① 1 is the unit
- ②  $e_3 = e_1e_2$
- ③  $e_1^2 \in \{-1, 0, 1\}$  and  $e_2^2 \in \{-1, 0, 1\}$
- ④ There exists  $\alpha \in \{-1, 1\}$  (hence  $\alpha^2 = 1$ ) such that, for  $1 \leq n, m \leq 3 : e_n(e_n e_m) = \alpha(e_n^2)e_m$  and  $(e_n e_m)e_m = \alpha e_n(e_m^2)$
- ⑤ There exists  $\gamma \in \{-1, 1\}$  (hence  $\gamma^2 = 1$ ) such that  $e_1e_2 = \gamma e_2e_1$ .

We can prove that  $e_1e_3 = \gamma e_3e_1$  (and  $e_2e_3 = \gamma e_3e_2$ ) :

$$\underline{e_1e_3 \stackrel{\textcircled{2}}{=} e_1(e_1e_2) \stackrel{\textcircled{4}}{=} \alpha(e_1^2)e_2 \stackrel{\textcircled{3}}{=} \alpha e_2(e_1^2) \stackrel{\textcircled{4}}{=} \alpha^2(e_2e_1)e_1 \stackrel{\textcircled{5}}{=} \gamma(e_1e_2)e_1 \stackrel{\textcircled{2}}{=} \gamma e_3e_1} \quad \text{QED}$$

This result will, also, be referred to as rule .

Such an algebra is fully described by the multiplication table of a basis, we will use  $[1, i, j, k]$ , rather than  $[1, e_1, e_2, e_3]$ .

The cells with a blank background are obtained by a straightforward application of the rules, the proofs for the cells with a blue background are given there : Proofs, and the cells with an orange background are obtained from the blue or blank ones applying only rule .

·	1	<i>i</i>	<i>j</i>	<i>k</i>
1	1	<i>i</i>	<i>j</i>	<i>k</i>
<i>i</i>	<i>i</i>	$i^2$	<i>k</i>	$\alpha i^2 j$
<i>j</i>	<i>j</i>	$\gamma k$	$j^2$	$\alpha \gamma j^2 i$
<i>k</i>	<i>k</i>	$\alpha \gamma i^2 j$	$\alpha j^2 i$	$\alpha \gamma j^2 i^2$

Generic  $\mathbb{R}^4$ -algebra

We will note such an algebra  $\mathbb{R}(1^2, j^2, \alpha, \gamma)$ .

At first glance there are  $3^2 \cdot 2^2 = 36$  different algebras of this type but as *i* and *j* play the same role we can impose  $i \leq j$  which gives 24 possibilities, some of them being isomorphic as seen in the following table.

In the following table,  $C\mathcal{D}(A, \lambda)$  is the result of the Cayley-Dickson construction from an algebra A with parameter  $\lambda$  defined by : Let  $B = A \oplus A$ , so B is a vector space whose dimension is  $2n$ , addition and scalar multiplication are naturally defined, but the multiplication and the involution ( $\bar{x}$  is the conjugate of *x*) must be specified :

$$(x, x') \times (y, y') = (xy - \lambda \bar{y}'x', y'x + x'\bar{y})$$

$$\overline{(x, x')} = (\bar{x}, -x')$$

$i^2$	$j^2$	$\alpha$	$\gamma$	Symbol	Name
-1	-1	-1	-1	ℳ	Split Hyperbolic Quaternions
-1	1	-1	-1		
-1	-1	1	-1	ℍ	Quaternions ( $\simeq \mathcal{CD}(\mathbb{C}, 1) \simeq \mathcal{Cl}(0, 2)$ )
-1	-1	1	1	ℂ <sub>2</sub>	BiComplex Numbers ( $\simeq \mathbb{C} \otimes \mathbb{C} \simeq \mathbb{C} \otimes \mathbb{C}$ )
-1	1	1	1		
-1	0	1	-1	$\mathcal{CD}(\mathbb{C}, 0)$	"Anticommutative" Dual Complex ( $\simeq \mathcal{CD}(\mathbb{D}_1, 1)$ )
-1	0	1	1	℔ <sub>2</sub>	Dual Complex ( $\simeq \mathbb{C} \otimes \mathbb{D}_1$ )
-1	1	1	-1	ℕ	CoQuaternions ( $\simeq \mathcal{Cl}(1, 1) \simeq \mathcal{Cl}(2, 0) \simeq \mathcal{CD}(\mathbb{C}, -1) \simeq \mathcal{CD}(\mathbb{C}, -1) \simeq \mathcal{CD}(\mathbb{C}, 1)$ )
1	1	1	-1		
0	0	-1	-1	Λ <sub>2</sub>	Grassmann Numbers on 2 generators ( $\simeq \mathcal{CD}(\mathbb{D}_1, 0)$ )
0	0	1	-1		
0	0	-1	1	℔ <sub>2</sub>	Hyper Dual Numbers ( $\simeq \mathbb{D}_1 \otimes \mathbb{D}_1$ )
0	0	1	1		
0	1	1	-1	$\mathcal{CD}(\mathbb{C}, 0)$	"Anticommutative" Dual Perplex ( $\simeq \mathcal{CD}(\mathbb{D}_1, -1)$ )
0	1	1	1	℔ <sub>2</sub> ⊗ ℔ <sub>1</sub>	Dual Perplex Numbers
1	1	-1	-1	ℳ	Hyperbolic Quaternions
1	1	1	1	℔ <sub>2</sub>	BiPerplex Numbers ( $\simeq \mathbb{C} \otimes \mathbb{C}$ )

## 4.2 Proofs for Dimension 4

The sign = alone means that the proof uses a previous result, or  $\alpha^2 = \gamma^2 = 1$ , or the fact that real numbers commute with every element.

The sign = with a circled number above means that the proof uses the rule bearing that same number.

$$\begin{array}{l}
 ik \stackrel{\textcircled{2}}{=} i(ij) \stackrel{\textcircled{4}}{=} \alpha(i^2)j \\
 ki \stackrel{\textcircled{5}}{=} \gamma ik = \alpha\gamma(i^2)j \\
 ji \stackrel{\textcircled{5}}{=} \gamma ij \stackrel{\textcircled{2}}{=} \gamma k \\
 jk \stackrel{\textcircled{2}}{=} j(ij) \stackrel{\textcircled{5}}{=} \gamma j(ji) \stackrel{\textcircled{4}}{=} \alpha\gamma(j^2)i \\
 kj \stackrel{\textcircled{5}}{=} \gamma jk = \gamma\alpha\gamma j^2 i = \alpha j^2 i \\
 (k^2)i \stackrel{\textcircled{4}}{=} \alpha k(ki) = k\gamma(i^2)j = \gamma\alpha j^2 i^2 \\
 \Rightarrow k^2 = \gamma\alpha j^2 i^2
 \end{array}$$

## 4.3 Multiplication Tables for Dimension 4

$\cdot$	1	$i$	$j$	$k$
1	1	$i$	$j$	$k$
$i$	$i$	-1	$k$	$j$
$j$	$j$	$-k$	-1	$-i$
$k$	$k$	$-j$	$i$	1

Split Hyperbolic Quaternions

$$\mathbb{R}^4(-1, \pm 1, -1, -1)$$

$\cdot$	1	$i$	$j$	$k$
1	1	$i$	$j$	$k$
$i$	$i$	-1	$k$	$-j$
$j$	$j$	$-k$	-1	$i$
$k$	$k$	$j$	$-i$	-1

Quaternions

$$\mathbb{R}^4(-1, -1, 1, -1)$$

$\cdot$	1	$i$	$j$	$k$
1	1	$i$	$j$	$k$
$i$	$i$	-1	$k$	$-j$
$j$	$j$	$k$	-1	$-i$
$k$	$k$	$-j$	$-i$	1

BiComplexes

$$\mathbb{R}^4(-1, \pm 1, 1, 1)$$

$\cdot$	1	$i$	$j$	$k$
1	1	$i$	$j$	$k$
$i$	$i$	-1	$k$	$-j$
$j$	$j$	$-k$	0	0
$k$	$k$	$j$	0	0

Anticommutative Dual Complex

$$\mathbb{R}^4(-1, 0, 1, -1)$$

$\cdot$	1	$i$	$j$	$k$
1	1	$i$	$j$	$k$
$i$	$i$	-1	$k$	$-j$
$j$	$j$	$k$	0	0
$k$	$k$	$-j$	0	0

Dual Complex

$$\mathbb{R}^4(-1, 0, 1, 1)$$

$\cdot$	1	$i$	$j$	$k$
1	1	$i$	$j$	$k$
$i$	$i$	-1	$k$	$-j$
$j$	$j$	$-k$	1	$-i$
$k$	$k$	$j$	$i$	1

CoQuaternions

$$\mathbb{R}^4(\pm 1, 1, 1, -1)$$

$\cdot$	1	$i$	$j$	$k$
1	1	$i$	$j$	$k$
$i$	$i$	0	$k$	0
$j$	$j$	$-k$	0	0
$k$	$k$	0	0	0

2-Grassmann

$$\mathbb{R}^4(0, 0, \pm 1, -1)$$

$\cdot$	1	$i$	$j$	$k$
1	1	$i$	$j$	$k$
$i$	$i$	0	$k$	0
$j$	$j$	$k$	0	0
$k$	$k$	0	0	0

2-Hyper-Dual Numbers

$$\mathbb{R}^4(0, 0, \pm 1, 1)$$

·	1	<i>i</i>	<i>j</i>	<i>k</i>
1	1	<i>i</i>	<i>j</i>	<i>k</i>
<i>i</i>	<i>i</i>	0	<i>k</i>	0
<i>j</i>	<i>j</i>	- <i>k</i>	1	- <i>i</i>
<i>k</i>	<i>k</i>	0	<i>i</i>	0

Anticommutative Dual Perplex  
 $\mathbb{R}^4(0, 1, 1, -1)$

·	1	<i>i</i>	<i>j</i>	<i>k</i>
1	1	<i>i</i>	<i>j</i>	<i>k</i>
<i>i</i>	<i>i</i>	0	<i>k</i>	0
<i>j</i>	<i>j</i>	<i>k</i>	1	<i>i</i>
<i>k</i>	<i>k</i>	0	<i>i</i>	0

Dual Perplex Numbers  
 $\mathbb{R}^4(0, 1, 1, 1)$

·	1	<i>i</i>	<i>j</i>	<i>k</i>
1	1	<i>i</i>	<i>j</i>	<i>k</i>
<i>i</i>	<i>i</i>	1	<i>k</i>	- <i>j</i>
<i>j</i>	<i>j</i>	- <i>k</i>	1	<i>i</i>
<i>k</i>	<i>k</i>	<i>j</i>	- <i>i</i>	1

Hyperbolic Quaternions  
 $\mathbb{R}^4(1, 1, -1, -1)$

·	1	<i>i</i>	<i>j</i>	<i>k</i>
1	1	<i>i</i>	<i>j</i>	<i>k</i>
<i>i</i>	<i>i</i>	1	<i>k</i>	<i>j</i>
<i>j</i>	<i>j</i>	<i>k</i>	1	<i>i</i>
<i>k</i>	<i>k</i>	<i>j</i>	<i>i</i>	1

BiPerplex Numbers  
 $\mathbb{R}^4(1, 1, 1, 1)$

### 5. Dimension 8

With dimension 8 we will need more parameters ...

Elements of a  $\mathbb{R}$ -algebra of dimension 8 can be written :  $z = a_0 + a_1 \bullet e_1 + a_2 \bullet e_2 + a_3 \bullet e_3 + a_4 \bullet e_4 + a_5 \bullet e_5 + a_6 \bullet e_6 + a_7 \bullet e_7$ , where  $a_i \in \mathbb{R}$ . For sake of simplicity we will not use the  $\bullet$  symbol anymore.

#### 5.1 Parametrization for Dimension 8

The following rules are also very easy to encode in a spreadsheet, to experiment the different possibilities and to create new algebras (particularly the algebras not mentioned here (there are many of them)), but beware of the existence of an isomorphism with well known algebra.

- ① 1 is the unit
- ②  $e_3 = e_1 e_2$
- ③  $e_1^2 \in \{-1, 0, 1\}$  and  $e_2^2 \in \{-1, 0, 1\}$
- ④ There exists  $\alpha \in \{-1, 1\}$  (hence  $\alpha^2 = 1$ ) such that, for  $1 \leq n, m \leq 7$  :  $e_n(e_n e_m) = \alpha(e_n^2)e_m$  and  $(e_n e_m)e_m = \alpha e_n(e_m^2)$
- ⑤ There exists  $\gamma \in \{-1, 1\}$  (hence  $\gamma^2 = 1$ ) such that  $e_1 e_2 = \gamma e_2 e_1$ .
- ⑥  $e_5 = e_1 e_4$ ,  $e_6 = e_2 e_4$ ,  $e_7 = e_3 e_4$ .
- ⑦  $e_4^2 \in \{-1, 0, 1\}$
- ⑧ There exists  $\beta \in \{-1, 1\}$  (hence  $\beta^2 = 1$ ) such that, for  $1 \leq n \leq 2$  :  $e_n e_4 = \beta e_4 e_n$ .
- ⑨ There exists  $\delta \in \{-1, 1\}$  (hence  $\delta^2 = 1$ ) such that, for  $1 \leq n, m \leq 3, n \neq m$  :  $e_n(e_m e_4) = \delta(e_n e_m)e_4$ ,  $(e_4 e_n)e_m = \delta e_4(e_n e_m)$ ,  $(e_n e_4)e_m = \delta e_n(e_4 e_m)$  and,  $(e_n e_4)(e_m e_4) = \delta e_n(e_4(e_m e_4))$ .

From rule ⑨, we can prove for  $1 \leq n < 3$ ,  $(e_n e_4)(e_m e_4) = \alpha \beta \delta e_4^2(e_n e_m)$  :

$$\overline{(e_n e_4)(e_m e_4) \stackrel{\textcircled{9}}{=} \delta e_n(e_4(e_m e_4)) \stackrel{\textcircled{8}}{=} \beta \delta e_n(e_4(e_4 e_m)) \stackrel{\textcircled{4}}{=} \beta \delta \alpha e_n(e_4^2 e_m) \stackrel{\textcircled{3}}{=} \alpha \beta \delta e_4^2(e_n e_m)}$$

QED

This result will, also, be referred to as rule ⑨.

Such an algebra is fully described by the multiplication table of a basis, we will use  $\langle 1, i, j, k, e_4, e_5, e_6, e_7 \rangle$ , rather than  $\langle 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle$ .

The cells with a blank background are obtained by a straightforward application of the rules, or the previous results in dimension 4; the proofs for the cells with a blue background are given there : **Proofs**, and the cells with an orange background are obtained from the blue or blank ones applying only rule ⑧.

	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$						
1	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$						
$i$	$i$	$i^2$	$k$	$\alpha i^2$	$j$	$e_5$	$\alpha i^2$	$e_4$	$\delta$	$e_7$	$\alpha \delta i^2$	$e_6$		
$j$	$j$	$\gamma$	$k$	$j^2$	$\alpha \gamma j^2$	$i$	$e_6$	$\gamma \delta$	$e_7$	$\alpha j^2$	$e_4$	$\alpha \gamma \delta j^2$	$e_5$	
$k$	$k$	$\alpha \gamma i^2$	$j$	$\alpha j^2$	$i$	$\alpha \gamma i^2 j^2$	$e_7$	$\alpha \gamma \delta i^2$	$e_6$	$\alpha \delta j^2$	$e_5$	$\gamma i^2 j^2$	$e_4$	
$e_4$	$e_4$	$\beta$	$e_5$	$\beta$	$e_6$	$\delta$	$e_7$	$e_4^2$	$\alpha \beta e_4^2$	$i$	$\alpha \beta e_4^2$	$j$	$\alpha \delta e_4^2$	$k$
$e_5$	$e_5$	$\alpha \beta i^2$	$e_4$	$\beta$	$e_7$	$\alpha \delta i^2$	$e_6$	$\alpha e_4^2$	$i$	$\alpha \beta i^2 e_4^2$	$\alpha \beta \delta e_4^2$	$k$	$i^2 e_4^2$	$j$
$e_6$	$e_6$	$\beta \gamma$	$e_7$	$\alpha \beta j^2$	$e_4$	$\alpha \gamma \delta j^2$	$e_5$	$\alpha e_4^2$	$j$	$\alpha \beta \delta \gamma e_4^2$	$k$	$\alpha \beta j^2 e_4^2$	$\gamma j^2 e_4^2$	$i$
$e_7$	$e_7$	$\alpha \beta \gamma i^2$	$e_6$	$\alpha \beta j^2$	$e_5$	$\gamma \delta i^2 j^2$	$e_4$	$\alpha e_4^2$	$k$	$\gamma \beta \delta i^2 e_4^2$	$j$	$\beta \delta j^2 e_4^2$	$i$	$\gamma \delta i^2 j^2 e_4^2$

Generic  $\mathbb{R}^8$ -algebra

We will note such an algebra  $\mathbb{R}^8(i^2, j^2, e_4^2, \alpha, \beta, \gamma, \delta)$ .

At first glance there are  $3^3 2^4 = 432$  different algebras of this type but as  $i, j$  and  $e_4$  play the same role we can impose  $i^2 \leq j^2 \leq e_4$ . which gives 160 possibilities, some (many) of them being isomorphic.

$i^2$	$j^2$	$e_4^2$	$\alpha$	$\beta$	$\gamma$	$\delta$	Symbol	Name
-1	-1	-1	1	-1	-1	-1	$\mathbb{O}$	Octonions
-1	-1	-1	1	1	-1	1	$\mathbb{B}$	BiQuaternions
-1	-1	-1	1	1	1	1	$\mathbb{C}_3$	TriComplex Numbers
-1	-1	0	1	1	-1	1	$\mathbb{D}_4$	Dual Quaternions
-1	-1	1	1	-1	-1	-1	$\mathbb{Q}$	Split Octonions
-1	-1	1	1	1	-1	1	$\mathbb{R}$	Split BiQuaternions
-1	1	0	1	1	-1	1	$\mathbb{D}_4$	Dual Split Quaternions
0	0	0	1	-1	-1	1	$\Lambda_3$	Grassmann Numbers
0	0	0	1	1	1	1	$\mathfrak{A}_3$	Hyper Dual Numbers
1	1	-1	1	-1	-1	1	$\mathcal{C}(2, 1)$	Clifford Algebra
1	1	1	1	-1	-1	1	$\mathcal{C}(3, 0)$	Clifford Algebra
1	1	1	1	1	1	1	$\mathbb{C}_3$	TriPerplex Numbers <sup>3</sup>

All the  $\mathbb{R}$ -algebra of dimension 8 that I know of can be described with these 7 parameters, but the Moreno Octonions (which would need an 8<sup>th</sup> parameter to split the incidence of  $\alpha$ ).

### 5.2 Proofs for Dimension 8

As in the case of dimension 4 :

The sign = alone means that the proof uses a previous result, or  $\alpha^2 = \gamma^2 = 1$ , or the fact that real numbers commute with every element.

The sign = with a circled number above it means that the proof uses the rule bearing that same number.

$ie_5$	$\stackrel{\textcircled{6}}{=} i(ie_4)$							$\stackrel{\textcircled{4}}{=} \alpha(i^2) e_4$
$ie_6$	$\stackrel{\textcircled{6}}{=} i(je_4)$	$\stackrel{\textcircled{9}}{=} \delta(ij)e_4$	$\stackrel{\textcircled{2}}{=} \delta ke_4$					$\stackrel{\textcircled{6}}{=} \delta e_7$
$ie_7$	$\stackrel{\textcircled{6}}{=} i(ke_4)$	$\stackrel{\textcircled{9}}{=} \delta(ik)e_4$	$= \delta\alpha i^2 je_4$					$\stackrel{\textcircled{6}}{=} \alpha\delta i^2 e_6$
$je_5$	$\stackrel{\textcircled{6}}{=} j(ie_4)$	$\stackrel{\textcircled{9}}{=} \delta(ji)e_4$	$= \delta\gamma ke_4$					$\stackrel{\textcircled{6}}{=} \gamma\delta e_7$
$je_6$	$\stackrel{\textcircled{6}}{=} j(je_4)$							$\stackrel{\textcircled{4}}{=} \alpha(j^2) e_4$
$je_7$	$\stackrel{\textcircled{6}}{=} j(ke_4)$	$\stackrel{\textcircled{9}}{=} \delta(jk)e_4$	$= \delta\gamma\alpha j^2 ie_4$					$\stackrel{\textcircled{6}}{=} \alpha\gamma\delta j^2 e_5$
$ke_5$	$\stackrel{\textcircled{6}}{=} k(ie_4)$	$\stackrel{\textcircled{9}}{=} \delta(ki)e_4$	$= \delta\alpha\gamma i^2 je_4$					$\stackrel{\textcircled{6}}{=} \alpha\gamma\delta i^2 e_6$
$ke_6$	$\stackrel{\textcircled{6}}{=} k(je_4)$	$\stackrel{\textcircled{9}}{=} \delta(kj)e_4$	$= \delta\alpha j^2 ie_4$					$\stackrel{\textcircled{6}}{=} \alpha\delta j^2 e_5$
$ke_7$	$\stackrel{\textcircled{6}}{=} k(ke_4)$	$\stackrel{\textcircled{4}}{=} \alpha(k^2)e_4$	$= \alpha\gamma\alpha i^2 j^2 e_4$					$= \gamma i^2 j^2 e_4$
$e_4 k$	$\stackrel{\textcircled{2}}{=} e_4(ij)$	$\stackrel{\textcircled{9}}{=} \delta(e_4 i)j$	$\stackrel{\textcircled{6}}{=} \delta\beta(i e_4)j$	$\stackrel{\textcircled{9}}{=} \beta i(e_4 j)$	$\stackrel{\textcircled{8}}{=} i(j e_4)$			$\stackrel{\textcircled{9}}{=} \delta e_7$
	$\stackrel{\textcircled{9}}{=} \delta(ij)e_4$	$\stackrel{\textcircled{2}}{=} \delta ke_4$						
$e_4 e_5$	$\stackrel{\textcircled{6}}{=} e_4(ie_4)$	$\stackrel{\textcircled{8}}{=} \beta e_4(e_4 i)$						$\stackrel{\textcircled{4}}{=} \alpha\beta e_4^2 i$
$e_4 e_6$	$\stackrel{\textcircled{6}}{=} e_4(je_4)$	$\stackrel{\textcircled{8}}{=} \beta e_4(e_4 j)$						$\stackrel{\textcircled{4}}{=} \alpha\beta e_4^2 j$
$e_4 e_7$	$\stackrel{\textcircled{6}}{=} e_4(ke_4)$	$= \delta e_4(e_4 k)$						$\stackrel{\textcircled{4}}{=} \alpha\delta e_4^2 k$
$e_5 i$	$\stackrel{\textcircled{6}}{=} (ie_4)i$	$\stackrel{\textcircled{8}}{=} \beta(e_4 i)i$						$\stackrel{\textcircled{4}}{=} \alpha\beta i^2 e_4$
$e_5 j$	$\stackrel{\textcircled{6}}{=} (ie_4)j$	$\stackrel{\textcircled{8}}{=} \beta(e_4 i)j$	$\stackrel{\textcircled{9}}{=} \beta\delta e_4(ij)$	$\stackrel{\textcircled{2}}{=} \beta\delta e_4 k$	$= \beta ke_4$			$\stackrel{\textcircled{6}}{=} \beta e_7$
$e_5 k$	$\stackrel{\textcircled{6}}{=} (ie_4)k$	$\stackrel{\textcircled{8}}{=} \beta(e_4 i)k$	$\stackrel{\textcircled{9}}{=} \beta\delta e_4(ik)$	$= \beta\delta\alpha i^2 e_4 j$	$\stackrel{\textcircled{8}}{=} \delta\alpha i^2 je_4$			$\stackrel{\textcircled{6}}{=} \alpha\delta i^2 e_6$
$e_5 e_4$	$\stackrel{\textcircled{6}}{=} (ie_4)e_4$							$\stackrel{\textcircled{4}}{=} \alpha e_4^2 i$
$(e_5^2)i$	$\stackrel{\textcircled{4}}{=} \alpha e_5(e_5 i)$	$= \beta i^2 e_5 e_4$	$= \alpha\beta i^2 e_4^2 i$		$\Rightarrow$	$e_5^2$	$= \alpha\beta i^2 e_4^2 i$	
$e_5 e_6$	$\stackrel{\textcircled{6}}{=} (ie_4)(je_4)$	$\stackrel{\textcircled{9}}{=} \alpha\beta\delta e_4^2(ij)$						$\stackrel{\textcircled{2}}{=} \alpha\beta\delta e_4^2 k$
$e_5 e_7$	$\stackrel{\textcircled{6}}{=} (ie_4)(ke_4)$	$\stackrel{\textcircled{9}}{=} \delta i(e_4(ke_4))$	$\stackrel{\textcircled{6}}{=} \delta i(e_4 e_7)$	$= \delta i(\alpha\delta e_4^2 k)$	$= \alpha e_4^2(\alpha(i^2)j)$			$= e_4^2 i^2 j$
$e_6 i$	$\stackrel{\textcircled{6}}{=} (je_4)i$	$\stackrel{\textcircled{8}}{=} \beta(e_4 j)i$	$\stackrel{\textcircled{9}}{=} \beta\delta e_4(ji)$	$= \beta\delta\gamma e_4 k$				$= \beta\gamma e_7$
$e_6 j$	$\stackrel{\textcircled{6}}{=} (je_4)j$	$\stackrel{\textcircled{8}}{=} \beta(e_4 j)j$						$\stackrel{\textcircled{4}}{=} \alpha\beta j^2 e_4$
$e_6 k$	$\stackrel{\textcircled{6}}{=} (je_4)k$	$\stackrel{\textcircled{8}}{=} \beta(e_4 j)k$	$\stackrel{\textcircled{9}}{=} \beta\delta e_4(jk)$	$= \alpha\gamma\beta\delta j^2 e_4 i$	$\stackrel{\textcircled{8}}{=} \alpha\gamma\delta j^2 ie_4$			$\stackrel{\textcircled{6}}{=} \alpha\gamma\delta j^2 e_5$
$e_6 e_4$	$\stackrel{\textcircled{6}}{=} (je_4)e_4$							$\stackrel{\textcircled{4}}{=} \alpha e_4^2 j$
$e_6 e_5$	$\stackrel{\textcircled{6}}{=} (je_4)(ie_4)$	$\stackrel{\textcircled{9}}{=} \alpha\beta\delta e_4^2(ji)$						$= \alpha\beta\delta e_4^2 \gamma k$
$(e_6^2)j$	$\stackrel{\textcircled{4}}{=} \alpha e_6(e_6 j)$	$= \beta j^2 e_6 e_4$	$= \beta j^2 \alpha e_4^2 j$		$\Rightarrow$	$e_6^2$	$= \alpha\beta j^2 e_4^2 j$	
$e_6 e_7$	$\stackrel{\textcircled{6}}{=} (je_4)(ke_4)$	$\stackrel{\textcircled{9}}{=} \delta j(e_4(ke_4))$	$\stackrel{\textcircled{6}}{=} \delta j(e_4 e_7)$	$= \delta j(\alpha\delta e_4^2 k)$				$= \gamma j^2 e_4^2 i$
$e_7 i$	$\stackrel{\textcircled{6}}{=} (ke_4)i$	$= \delta(e_4 k)i$	$\stackrel{\textcircled{9}}{=} e_4(ki)$	$= \alpha\gamma i^2 e_4 j$	$= \alpha\gamma\beta i^2 je_4$			$\stackrel{\textcircled{6}}{=} \alpha\beta\gamma i^2 e_6$
$e_7 j$	$\stackrel{\textcircled{6}}{=} (ke_4)j$	$= \delta(e_4 k)j$	$\stackrel{\textcircled{9}}{=} e_4(kj)$	$= \alpha j^2 e_4 i$	$= \alpha\beta j^2 ie_4$			$\stackrel{\textcircled{6}}{=} \alpha\beta j^2 e_5$
$e_7 k$	$\stackrel{\textcircled{6}}{=} (ke_4)k$	$= \delta(e_4 k)k$	$\stackrel{\textcircled{4}}{=} \alpha\delta e_4 k^2$					$= \gamma\delta j^2 i^2 e_4$
$e_7 e_4$	$\stackrel{\textcircled{6}}{=} (ke_4)e_4$							$\stackrel{\textcircled{4}}{=} \alpha e_4^2 k$
$e_7 e_5$	$\stackrel{\textcircled{6}}{=} (ke_4)(ie_4)$	$\stackrel{\textcircled{9}}{=} \delta k(e_4(ie_4))$	$\stackrel{\textcircled{6}}{=} \delta k(e_4 e_5)$	$= \delta k(\alpha\beta e_4^2 i)$	$= \alpha\beta\delta e_4^2 ki$			$= \beta\gamma\delta i^2 e_4^2 j$
$e_7 e_6$	$\stackrel{\textcircled{6}}{=} (ke_4)(je_4)$	$\stackrel{\textcircled{9}}{=} \delta k(e_4(je_4))$	$\stackrel{\textcircled{6}}{=} \delta k(e_4 e_6)$	$= \delta k(\alpha\beta e_4^2 j)$	$= \alpha\beta\delta e_4^2 kj$			$= \beta\delta e_4^2 j^2 i$
$(e_7^2)k$	$\stackrel{\textcircled{4}}{=} \alpha e_7(e_7 k)$	$= \alpha\delta\gamma j^2 i^2 e_7 e_4$	$= \delta\gamma j^2 i^2 e_4^2 k$		$\Rightarrow$	$e_7^2$	$= \gamma\delta j^2 i^2 e_4^2 k$	

### 5.3 Multiplication Tables for Dimension 8

$\cdot$	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
1	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
$i$	$i$	-1	$k$	$-j$	$e_5$	$-e_4$	$-e_7$	$e_6$
$j$	$j$	$-k$	-1	$i$	$e_6$	$e_7$	$-e_4$	$-e_5$
$k$	$k$	$j$	$-i$	-1	$e_7$	$-e_6$	$e_5$	$-e_4$
$e_4$	$e_4$	$-e_5$	$-e_6$	$-e_7$	-1	$i$	$j$	$k$
$e_5$	$e_5$	$e_4$	$-e_7$	$e_6$	$-i$	-1	$-k$	$j$
$e_6$	$e_6$	$e_7$	$e_4$	$-e_5$	$-j$	$k$	-1	$-i$
$e_7$	$e_7$	$-e_6$	$e_5$	$e_4$	$-k$	$-j$	$i$	-1

Octonions

$$\mathbb{R}^8(-1, -1, -1, 1, -1, -1, -1)$$

$\cdot$	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
1	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
$i$	$i$	-1	$k$	$-j$	$e_5$	$-e_4$	$e_7$	$-e_6$
$j$	$j$	$-k$	-1	$i$	$e_6$	$-e_7$	$-e_4$	$e_5$
$k$	$k$	$j$	$-i$	-1	$e_7$	$e_6$	$-e_5$	$-e_4$
$e_4$	$e_4$	$e_5$	$e_6$	$e_7$	-1	$-i$	$-j$	$-k$
$e_5$	$e_5$	$-e_4$	$e_7$	$-e_6$	$-i$	1	$-k$	$j$
$e_6$	$e_6$	$-e_7$	$-e_4$	$e_5$	$-j$	$k$	1	$-i$
$e_7$	$e_7$	$e_6$	$-e_5$	$-e_4$	$-k$	$-j$	$i$	1

BiQuaternions  
 $\mathbb{R}^8(-1, -1, -1, 1, 1, -1, 1)$

$\cdot$	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
1	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
$i$	$i$	-1	$k$	$-j$	$e_5$	$-e_4$	$e_7$	$-e_6$
$j$	$j$	$k$	-1	$-i$	$e_6$	$e_7$	$-e_4$	$-e_5$
$k$	$k$	$-j$	$-i$	1	$e_7$	$-e_6$	$-e_5$	$e_4$
$e_4$	$e_4$	$e_5$	$e_6$	$e_7$	-1	$-i$	$-j$	$-k$
$e_5$	$e_5$	$-e_4$	$e_7$	$-e_6$	$-i$	1	$-k$	$j$
$e_6$	$e_6$	$e_7$	$-e_4$	$-e_5$	$-j$	$-k$	1	$i$
$e_7$	$e_7$	$-e_6$	$-e_5$	$e_4$	$-k$	$j$	$i$	-1

Tricomplex Numbers  
 $\mathbb{R}^8(-1, -1, -1, 1, 1, 1, 1)$

$\cdot$	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
1	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
$i$	$i$	-1	$k$	$-j$	$e_5$	$-e_4$	$e_7$	$-e_6$
$j$	$j$	$-k$	-1	$i$	$e_6$	$-e_7$	$-e_4$	$e_5$
$k$	$k$	$j$	$-i$	-1	$e_7$	$e_6$	$-e_5$	$-e_4$
$e_4$	$e_4$	$e_5$	$e_6$	$e_7$	0	0	0	0
$e_5$	$e_5$	$-e_4$	$e_7$	$-e_6$	0	0	0	0
$e_6$	$e_6$	$-e_7$	$-e_4$	$e_5$	0	0	0	0
$e_7$	$e_7$	$e_6$	$-e_5$	$-e_4$	0	0	0	0

Dual Quaternions  
 $\mathbb{R}^8(-1, -1, 0, 1, 1, -1, 1)$

$\cdot$	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
1	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
$i$	$i$	-1	$k$	$-j$	$e_5$	$-e_4$	$-e_7$	$e_6$
$j$	$j$	$-k$	-1	$i$	$e_6$	$e_7$	$-e_4$	$-e_5$
$k$	$k$	$j$	$-i$	-1	$e_7$	$-e_6$	$e_5$	$-e_4$
$e_4$	$e_4$	$-e_5$	$-e_6$	$-e_7$	1	$-i$	$-j$	$-k$
$e_5$	$e_5$	$e_4$	$-e_7$	$e_6$	$i$	1	$k$	$-j$
$e_6$	$e_6$	$e_7$	$e_4$	$-e_5$	$j$	$-k$	1	$i$
$e_7$	$e_7$	$-e_6$	$e_5$	$e_4$	$k$	$j$	$-i$	1

Split Octonions  
 $\mathbb{R}^8(-1, -1, 1, 1, -1, -1, -1)$

$\cdot$	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
1	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
$i$	$i$	-1	$k$	$-j$	$e_5$	$-e_4$	$e_7$	$-e_6$
$j$	$j$	$-k$	-1	$i$	$e_6$	$-e_7$	$-e_4$	$e_5$
$k$	$k$	$j$	$-i$	-1	$e_7$	$e_6$	$-e_5$	$-e_4$
$e_4$	$e_4$	$e_5$	$e_6$	$e_7$	1	$i$	$j$	$k$
$e_5$	$e_5$	$-e_4$	$e_7$	$-e_6$	$i$	-1	$k$	$-j$
$e_6$	$e_6$	$-e_7$	$-e_4$	$e_5$	$j$	$-k$	-1	$i$
$e_7$	$e_7$	$e_6$	$-e_5$	$-e_4$	$k$	$j$	$-i$	-1

Split BiQuaternions  
 $\mathbb{R}^8(-1, -1, 1, 1, 1, -1, 1)$

$\cdot$	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
1	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
$i$	$i$	-1	$k$	$-j$	$e_5$	$-e_4$	$e_7$	$-e_6$
$j$	$j$	$-k$	1	$-i$	$e_6$	$-e_7$	$e_4$	$-e_5$
$k$	$k$	$j$	$i$	1	$e_7$	$e_6$	$e_5$	$e_4$
$e_4$	$e_4$	$e_5$	$e_6$	$e_7$	0	0	0	0
$e_5$	$e_5$	$-e_4$	$e_7$	$-e_6$	0	0	0	0
$e_6$	$e_6$	$-e_7$	$e_4$	$-e_5$	0	0	0	0
$e_7$	$e_7$	$e_6$	$e_5$	$e_4$	0	0	0	0

Dual Split Quaternions  
 $\mathbb{R}^8(-1, 1, 0, 1, 1, -1, 1)$

$\cdot$	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
1	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
$i$	$i$	0	$k$	0	$e_5$	0	$e_7$	0
$j$	$j$	$-k$	0	0	$e_6$	$-e_7$	0	0
$k$	$k$	0	0	0	$e_7$	0	0	0
$e_4$	$e_4$	$-e_5$	$-e_6$	$e_7$	0	0	0	0
$e_5$	$e_5$	0	$-e_7$	0	0	0	0	0
$e_6$	$e_6$	$e_7$	0	0	0	0	0	0
$e_7$	$e_7$	0	0	0	0	0	0	0

3-Grassmann Numbers  
 $\mathbb{R}^8(0, 0, 0, 1, -1, -1, 1)$

$\cdot$	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
1	1	$i$	$j$	$k$	$e_4$	$e_5$	$e_6$	$e_7$
$i$	$i$	0	$k$	0	$e_5$	0	$e_7$	0
$j$	$j$	$k$	0	0	$e_6$	$e_7$	0	0
$k$	$k$	0	0	0	$e_7$	0	0	0
$e_4$	$e_4$	$e_5$	$e_6$	$e_7$	0	0	0	0
$e_5$	$e_5$	0	$e_7$	0	0	0	0	0
$e_6$	$e_6$	$e_7$	0	0	0	0	0	0
$e_7$	$e_7$	0	0	0	0	0	0	0

3-Hyper Dual Numbers  
 $\mathbb{R}^8(0, 0, 0, 1, 1, 1, 1)$



·	1	<i>i</i>	<i>j</i>	<i>k</i>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>
1	1	<i>i</i>	<i>j</i>	<i>k</i>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>
<i>i</i>	<i>i</i>	1	<i>k</i>	<i>j</i>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>7</sub>	<i>e</i> <sub>6</sub>
<i>j</i>	<i>j</i>	- <i>k</i>	1	- <i>i</i>	<i>e</i> <sub>6</sub>	- <i>e</i> <sub>7</sub>	<i>e</i> <sub>4</sub>	- <i>e</i> <sub>5</sub>
<i>k</i>	<i>k</i>	- <i>j</i>	<i>i</i>	-1	<i>e</i> <sub>7</sub>	- <i>e</i> <sub>6</sub>	<i>e</i> <sub>5</sub>	- <i>e</i> <sub>4</sub>
<i>e</i> <sub>4</sub>	<i>e</i> <sub>4</sub>	- <i>e</i> <sub>5</sub>	- <i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>	-1	<i>i</i>	<i>j</i>	- <i>k</i>
<i>e</i> <sub>5</sub>	<i>e</i> <sub>5</sub>	- <i>e</i> <sub>4</sub>	- <i>e</i> <sub>7</sub>	<i>e</i> <sub>6</sub>	- <i>i</i>	1	<i>k</i>	- <i>j</i>
<i>e</i> <sub>6</sub>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>	- <i>e</i> <sub>4</sub>	- <i>e</i> <sub>5</sub>	- <i>j</i>	- <i>k</i>	1	<i>i</i>
<i>e</i> <sub>7</sub>	<i>e</i> <sub>7</sub>	<i>e</i> <sub>6</sub>	- <i>e</i> <sub>5</sub>	- <i>e</i> <sub>4</sub>	- <i>k</i>	- <i>j</i>	<i>i</i>	1

Clifford Algebra  $Cl(2, 1)$   
 $\mathbb{R}^8(1, 1, -1, 1, -1, -1, 1)$

·	1	<i>i</i>	<i>j</i>	<i>k</i>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>
1	1	<i>i</i>	<i>j</i>	<i>k</i>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>
<i>i</i>	<i>i</i>	1	<i>k</i>	<i>j</i>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>7</sub>	<i>e</i> <sub>6</sub>
<i>j</i>	<i>j</i>	- <i>k</i>	1	- <i>i</i>	<i>e</i> <sub>6</sub>	- <i>e</i> <sub>7</sub>	<i>e</i> <sub>4</sub>	- <i>e</i> <sub>5</sub>
<i>k</i>	<i>k</i>	- <i>j</i>	<i>i</i>	-1	<i>e</i> <sub>7</sub>	- <i>e</i> <sub>6</sub>	<i>e</i> <sub>5</sub>	- <i>e</i> <sub>4</sub>
<i>e</i> <sub>4</sub>	<i>e</i> <sub>4</sub>	- <i>e</i> <sub>5</sub>	- <i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>	1	- <i>i</i>	- <i>j</i>	<i>k</i>
<i>e</i> <sub>5</sub>	<i>e</i> <sub>5</sub>	- <i>e</i> <sub>4</sub>	- <i>e</i> <sub>7</sub>	<i>e</i> <sub>6</sub>	<i>i</i>	-1	- <i>k</i>	<i>j</i>
<i>e</i> <sub>6</sub>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>	- <i>e</i> <sub>4</sub>	- <i>e</i> <sub>5</sub>	<i>j</i>	<i>k</i>	-1	- <i>i</i>
<i>e</i> <sub>7</sub>	<i>e</i> <sub>7</sub>	<i>e</i> <sub>6</sub>	- <i>e</i> <sub>5</sub>	- <i>e</i> <sub>4</sub>	<i>k</i>	<i>j</i>	- <i>i</i>	-1

Clifford Algebra  $Cl(3, 0)$   
 $\mathbb{R}^8(1, 1, 1, 1, -1, -1, 1)$

·	1	<i>i</i>	<i>j</i>	<i>k</i>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>
1	1	<i>i</i>	<i>j</i>	<i>k</i>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>
<i>i</i>	<i>i</i>	1	<i>k</i>	<i>j</i>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>7</sub>	<i>e</i> <sub>6</sub>
<i>j</i>	<i>j</i>	<i>k</i>	1	<i>i</i>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>5</sub>
<i>k</i>	<i>k</i>	<i>j</i>	<i>i</i>	1	<i>e</i> <sub>7</sub>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>4</sub>
<i>e</i> <sub>4</sub>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>	1	<i>i</i>	<i>j</i>	<i>k</i>
<i>e</i> <sub>5</sub>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>7</sub>	<i>e</i> <sub>6</sub>	<i>i</i>	1	<i>k</i>	<i>j</i>
<i>e</i> <sub>6</sub>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>7</sub>	<i>e</i> <sub>4</sub>	<i>e</i> <sub>5</sub>	<i>j</i>	<i>k</i>	1	<i>i</i>
<i>e</i> <sub>7</sub>	<i>e</i> <sub>7</sub>	<i>e</i> <sub>6</sub>	<i>e</i> <sub>5</sub>	<i>e</i> <sub>4</sub>	<i>k</i>	<i>j</i>	<i>i</i>	1

Proto-Octonions  
 $\mathbb{R}^8(1, 1, 1, 1, 1, 1, 1)$

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