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On Rose Curves in the Polar Coordinate System

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Abstract
The article examines rose curves in the polar coordinate system, where each rose curve is determined by a cosine function with arbitrary positive amplitude and angular frequency. The amplitude refers to the radius of a circle whose center is the pole in which a rose curve is completely inscribed, and the angular frequency refers to the number of petals of a rose curve. Depending on the values of the angular frequency, which can be an integer, a rational number in the form of an irreducible fraction or an irrational number, the number of petals of a rose curve, the length of the interval for which a rose curve is complete and the polar angle between the peaks of the successive petals of a rose curve are examined. All mathematical considerations are accompanied by suitable examples and pictures.

Keywords: Angular Frequency, Radius, Polar Angle, Petal

1. Introduction

This article examines and describes a rose curve in more detail. The special feature of a rose curve is its shape, which resembles the shape of a flower with petals. It can therefore be associated in nature with many flowers with petals (e.g. asters, daisies, composite flowers or sunflowers), but also with leaves. The rose curve is not only associated with shapes in nature, but is also used in art, especially in architecture, where it is used as a decorative element in the form of a rose blossom with stylized petals. It can be found on many church facades, where it is used as a decorative window. Figure 1shows the Church of St. Mary of a Benedictine monastery in Zadar, whose facade is decorated with rose windows.

It should be noted that rose arches are also found in the cuisine of many cultures, known as rosettes. They are made with a rosette iron, which has a long handle with flowers in metal shapes. Rosettes are crisp and are characterized by their lace-like pattern. They are typical of Anglo-Indian cuisine and there are many versions of this cookie in Northern Europe, Tunis, the Middle East and West Asia and other places under different names. Rosettes are also popular with families in many countries and are traditionally baked at Christmas time.In mathematics, a rose curve has been studied by many researchers and mathematicians. One of them is Luigi Guido Grandi, who is known for his definition of the rodonea curve. Rodonea is the Latin word for rose, and Grandi was the first to define this curve [1-4].

Figure 1: The Church of St. Mary of a Benedictine Monastery in Zadar, Croatia

We therefore briefly describe his biography, part of which we quote from [2]. Grandi was a Camaldolese monk, philosopher, mathematician and engineer who worked in geometry and hydraulics. He was born on October 1, 1671 in Cremona and died on July 4, 1742 in Pisa. In 1694, Grandi became a teacher of philosophy and theology at the Camaldolese monastery of Santa Maria degli Angeli in Florence. Until then, he had shown little interest in mathematics, but then he turned to mathematics and studied the works of Euclid, Apollonius, Pappus and Archimedes. He learned the methods of classical geometry from Vincenzo Viviani and his students as well as the infinitesimal methods of Bonaventura Cavalieri. In 1699 he published *Geometrica divinatio Vivianeorum problematum*. After learning a great deal about geometry, he began to research applications for optics, mechanics and astronomy. In 1700, Grandi began teaching mathematics at the monastery of Santa Maria degli Angeli in Florence. Grandi began to work more intensively on mathematics and exchanged letters with many scientists and theologians. In 1703, he published the book *Quadratura circoli et hyperbolae per infinitas hyperbolas et parabolas quadrabiles geometrice exhibita*, which contained nothing particularly original but was important for the introduction of the infinitesimal method in Italy. In Italy, he was the first to teach the infinitesimal methods of calculus in private lessons. Grandi had studied Newton's fluxions and Leibniz's differentials and used both approaches, although he preferred Leibniz's approach. Grandi was appointed professor of mathematics at the University of Pisa in 1714. As already mentioned, one of the results for which Grandi is best known today is his definition of the Rodonea curve, i.e. the rose curve. He first defined these curves in December 1713 in a letter he wrote to Leibniz. He only published his results on these curves ten years later in the Philosophical Transactions of the Royal Society of London under the title *Handful or bouquet of geometrical roses*. In 1728, Grandi expanded the material on these curves in F*lores geometrici ex Rhodonearum, et Cloeliarum curvarum descriptione resultantes.* According to this, a rose curve is a curve in the plane that has the shape of a flower and is also known as Grandi's rose or multifolium. Folium is the Latin word for leaf and refers to a curve with leaf-shaped, rounded lobes.

2. The Polar Coordinate System

Since we are dealing in more detail with rose curves in the polar coordinate system, we will first briefly explain the properties of the polar coordinate system, the description of a point in polar coordinates depending on the sign of the radius and the polar angle and then the graphs of a function determined by a polar equation. The polar coordinate system is a two-dimensional coordinate system defined by a reference point, the *pole*, and the ray emanating from the pole in the reference direction, the *polar axis*. We denote the pole by P and the polar axis by p. In contrast to the Cartesian plane (or the rectangular coordinate system in the plane), which is defined by a reference point O, the origin, and two perpendicular reference lines, the x-axis and the y-axis, whose intersection is the origin, where the coordinates (x_0,y_0) of a point P_0 are determined by the intersection of two lines $x = x_0$ and $y = y_0$ (the first line is perpendicular to the x-axis, the other perpendicular to the y-axis), the polar coordinate system is a system of circles whose center is the pole P and rays emanating from P. We note that the pole is analogous to the origin of the Cartesian coordinate system in the plane and the polar axis can be considered analogous to the positive x-axis of the Cartesian plane.

Each point T in the polar coordinate system is determined by the intersection of the circle with the radius r, whose center is the pole P, and the ray PT emanating from the pole P, which determines the polar angle ϕ with the polar axis. The distance between the point T and the pole P is a positive real number that is equal to the radius r of the circle. A point in the polar coordinate system is therefore determined by a distance from the pole and an angle to the polar axis, so that the polar coordinates of a point T in the polar coordinate system are given by the ordered pair (r,ϕ), where r is the radius (or the radial coordinate or the radial distance), and ϕ is the polar angle (or the angular coordinate or the azimuth). The two r and ϕ are referred to as the polar coordinates of a point P in the polar coordinate system.

The measure of the polar angle ϕ can be expressed either in degrees or radians. In this article, we will only use radians. Recall that the conversion from radians to degrees is given by the relationship that π radians equals 180 degrees. In accordance with the fact that radius and polar angle are constants that can have a positive or negative sign, we have the following cases.

1. A polar angle ϕ is measured counterclockwise from the polar axis if $\phi > 0$, while it is measured clockwise if $\phi < 0$.
2. A point (r,ϕ) with $r > 0$ is measured for r units along the ray that determines the polar angle ϕ with the polar axis.
3. A point (r,ϕ) with $r < 0$ is measured for $|r|$ units along the ray that determines the polar angle $\phi+\pi$ with the polar *axis*, where $|r|$ denotes the absolute value of r.

We note that in contrast to the origin of the Cartesian coordinate system, which is uniquely determined by the point (0,0), the pole in the polar coordinate system is determined by an infinite number of points $(0,\phi)$, where ϕ is any polar angle. The second and third properties imply

$$(r,\phi) = (-r,\phi + \pi), \tag{1}$$

which means that the point (r,ϕ) with a positive radius is identical to the point whose polar coordinates consist of the corresponding negative radius and the opposite direction of the polar angle ϕ. In contrast to points in the Cartesian plane, where each point is uniquely determined by its coordinates, a point (r,ϕ) in the polar coordinate system has an infinite number of representations, which are written in the following form

$$(r,\phi) = (r,\phi + 2l\pi), \tag{2}$$

where l is an integer, see [1]. The following identity $(r,\phi) = (-r,\phi+(2l+1)\pi)$ results from (1) and (2), where l is an integer. In this way, the radius r can be restricted to any positive real number and the polar angle ϕ can be restricted to the interval $[0,2\pi)$, since the line $\phi = 2\pi$ is identical to the line $\phi = 0$ (the polar axis). Due to the uniqueness of the pole, it is common to choose the polar coordinates $r = 0$ and $\phi = 0$ for the pole.

Example 1. *Two points in the polar coordinate system are shown on the left-hand side of Figure 2. One is the point* $(\frac{3}{2}, \frac{\pi}{6})$ *whose radius is equal to* $\frac{3}{2}$ *and polar angle is equal to* $\frac{\pi}{6}$, *and the other is the point* $(2, \frac{2\pi}{3})$ *whose radius is equal to 2 and polar angle is equal to* $\frac{2\pi}{3}$. *The right-hand side of Figure 2 also shows two points with the corresponding positive and negative radius, whose polar angles run in opposite directions.*

A curve in the polar coordinate system is the locus of all points $(r;\phi)$ that are the solution of a polar equation expressed in the form

$$r = r(\phi), \qquad (3)$$

where the radius is defined as a real-valued function of a real variable. In other words, the value of the radius r is a real number that depends on the independent variable (polar angle) ϕ.

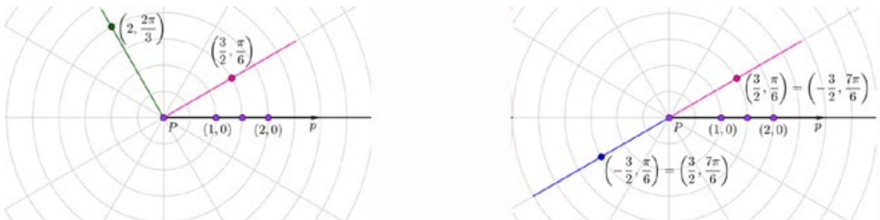

Figure 2: Points in the Polar Coordinate System

Therefore the curve determined by (3) consists of all points of the form $(r(\phi),\phi)$ that fulfil the three properties mentioned above. A curve determined by the polar equation (3) is symmetric with respect to the polar axis if replacing $(r;\phi)$ by $(r;-\phi)$ leads to the same equation, while it is symmetric with respect to the pole if replacing $(r;\phi)$ by $(-r;\phi)$ leads to the same equation. Similarly, it is symmetrical with respect to the line $\phi = \frac{\pi}{2}$ if replacing $(r;\phi)$ with $(r;\pi-\phi)$ leads to the same equation.

3. A Rose Curve

In general, the polar equation of rose curves is given by

$$r = a\cos k\phi \qquad (4)$$

or $r = a\sin k\phi$, where $a \neq 0$ and $k \neq 0$ are non-zero constants. Recall that the general forms of the sine and cosine functions in the Cartesian plane are $y = a\sin(bx-c)+d$ and $y = a\cos(bx - c) + d$, or

$$y = a\sin\left(b\left(x - \frac{c}{b}\right)\right) + d \text{ and } \quad y = a\cos\left(b\left(x - \frac{c}{b}\right)\right) + d \quad (5)$$

where $a \neq 0$, $b \neq 0$, c and d are constants. The domain of each of these two functions is a set of real numbers $\mathbb{R} = \langle -\infty, \infty \rangle$ and the range is an interval $[-||a||,||a||] \subset \mathbb{R}$, where $||a||$ denotes the absolute values of a constant $a \in \mathbb{R}\backslash\{0\}$ and is referred to as *amplitude* or greatest distance from rest. A constant $b \in \mathbb{R}\backslash\{0\}$ is related to the period by $P = \frac{2\pi}{||b||}$, where if $||b||>1$, the period is less than 2π and the function undergoes horizontal compression, while if $||b||<1$, the period is greater than 2π and the function undergoes horizontal expansion.

The value $\frac{c}{b}$ is called the phase shift or horizontal shift of the basic sine or cosine function, where the graph shifts to the right when $\frac{c}{b}>0$ and the graph shifts to the left when $\frac{c}{b}<0$. A constant d is the *vertical shift* from the midline. In particular, the midline of each of the two functions (5) is at $y = d$. Since a periodic function is a function f for which a given horizontal shift h yields a function corresponding to the function f that we write as $f(x + h) = f(x)$ for all values of x in the domain of f, it follows that the sine and cosine functions are periodic functions with a period $P = \frac{2\pi}{|b|}$. For a simpler explanation, let us assume that $a = b = 1$ and $c = d = 0$. Then it follows from (5) that $y = \sin x$ and $y = \cos x$, i.e. 2π is the period of this sine and cosine function for which the identities $\sin x = \cos\left(x - \frac{\pi}{2}\right)$ and $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ are satisfied. In other words, the graph of the sine function is identical to the graph of the cosine function shifted to the right by $\frac{\pi}{2}$ units, and likewise the graph of the cosine function is identical to the graph of the sine $\frac{\pi}{2}$ function shifted to the left by units. These properties of the shifted graphs of the sine and cosine functions generally also apply under the assumption that the sine and cosine functions have the same amplitude, period, phase shift and vertical shift. In fact, the graph of the sine function and also the graph of the cosine function are usually referred to as sinusoids. In this way, we obtain in the polar coordinate system

$$a\sin k\phi = a\cos\left(k\phi - \frac{\pi}{2}\right) = a\cos\left(k\left(\phi - \frac{\pi}{2k}\right)\right), \quad (6)$$

which means that a rose curve $r = a\sin k\phi$ is identical to the rose curve $r = a\cos k\phi$ rotated counterclockwise by $\frac{\pi}{2k}$ radians, which leads us to the conclusion that the equation of the rose curve in the polar coordinate system can also be given by the sine function as $r = a\sin k\phi$ with $a \neq 0$ and $k \neq 0$. We note that the horizontal displacement in the Cartesian plane leads to a rotation in the polar coordinate system.

Example 2. *Figure 3 shows the rose curve* $r = 2\cos 3\phi$ *on the left and the rose curve* $r = 2\sin 3\phi$ *on the right. If we compare these two curves, we can see in Figure 3 that the rose curve* $r = 2\sin 3\phi$ *can also be obtained by the rose curve* $r = 2\cos 3\phi$ *rotated counterclockwise by* $\frac{\pi}{6}$ *radians, which is consistent with (6) for* $a = 2$ *and* $k = 3$. *In fact,* $2\sin 3\phi = 2\cos\left(3\left(\phi - \frac{\pi}{6}\right)\right)$.

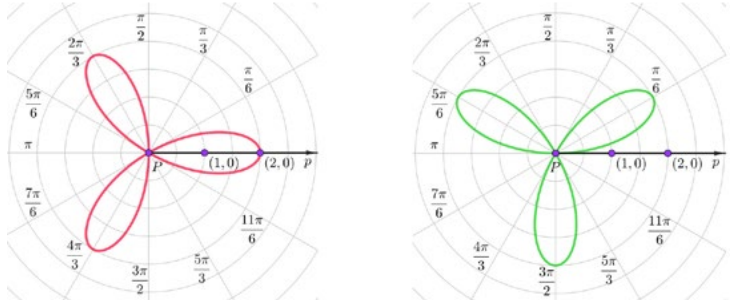

Figure 3: Rose Curves $r = 2\cos 3\phi$ and $r = 2\sin 3\phi$ In the Polar Coordinate System.

We emphasize that the non-zero constants a and ϕ, which refer to the cosine function in the equation (4) can be regarded as positive real numbers. Let us assume that a and k are positive constants. Since the cosine is an even function, i.e.

$$a\cos(-k\phi) = a\cos k\phi, \qquad (7)$$

the same rose curve is determined by two identical polar equations. On the other hand, the well-known trigonometric formula for cosine addition and cosine subtraction, i.e. $\cos(x \pm \pi) = -\cos x$, implies that

$$-a\cos k\phi = a\cos(k\phi \pm \pi) = a\cos\left(k\left(\phi \pm \frac{\pi}{k}\right)\right), \qquad (8)$$

which can be interpreted to mean that a rose curve $r = -a\cos k\phi$ is identical to a rose curve $r = a\cos k\phi$ that has been rotated around the pole by $\frac{\pi}{k}$.

In this way, it is sufficient to consider the non-zero constants $a \neq 0$ and $k \neq 0$ in the equation (4) as positive real numbers, where a constant a is the *amplitude* and k is the *angular frequency* of the cosine function. In the following, we therefore consider a rose curve in the polar coordinate system, which is determined by the equation (4), where a and k are positive real numbers (constants).

We note that the amplitude refers to the radius of a circle whose center is the pole in which a rose curve is completely inscribed, and the angular frequency refers to the number of petals of a rose curve. In fact, each rose curve consists of petals and each petal has a peak that lies on the circle.

From the property that a curve determined by the polar equation (3) is symmetric with respect to the polar axis if replacing (r,ϕ) by $(r,-\phi)$ leads to the same equation, it follows from the identity (7) that a rose curve determined by (4) is symmetric with respect to the polar axis. On the other hand, it follows from the identity (8) that the rose curves $r = -a\cos k\phi$ and $r = a\cos k\phi$ are also symmetrical with respect to the line $\varphi = \frac{\pi}{2}$.

Example 3. *Figure 4 shows three rose curves determined by the equation (4) such that the angular frequency of the cosine function is the same for all three rose curves, but the amplitude of the cosine function is one for one of them and the same for the other two with different signs. The left side of Figure 4 shows the rose curve $r = \cos 3\phi$ in purple and the rose curve $r = 2\cos 3\phi$ in red. On the right-hand side of Figure 4, the rose curve $r = -2\cos 3\phi$ is shown in blue. The rose curves $r = 2\cos 3\phi$ and $r = -2\cos 3\phi$ are symmetrical with respect to the line $\phi = pi/2$* .

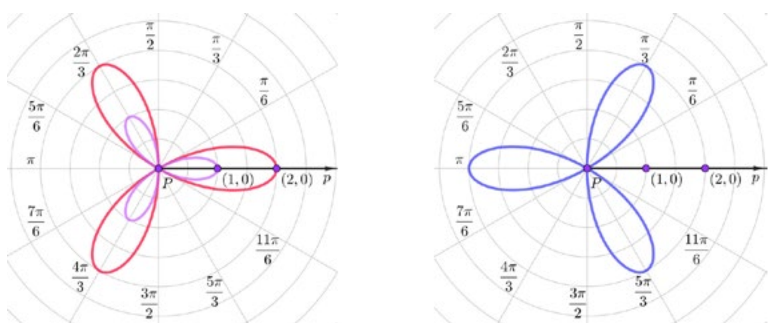

Figure 4: Rose curves $r = \cos 3\phi$, $r = 2\cos 3\phi$ and $r = -2\cos 3\phi$.

Furthermore, due to (8), the rose curve $r = -2\cos 3\phi$ is identical to $r = 2\cos 3\phi$ rotated by $\frac{\pi}{3}$ radians.

With the motivation to find a better explanation for the properties of rose curves, we discuss below a positive angular frequency of the cosine function in the equation (4), which corresponds to a positive integer or a positive rational or a positive irrational angular frequency, see [3]. Based on the fact that every integer is a special case of the rational number whose denominator is equal to one, we will also explain that a rose curve determined by the equation (4) is complete if the angular frequency of the cosine function is a positive rational number, otherwise a rose curve is incomplete. In other words, a rose curve is complete if the angular frequency of the cosine function is any positive integer or rational number. Then the corresponding interval of polar angles can be any continuous interval that is a proper subset of the set of real numbers. In the case of an irrational angular frequency, a rose curve is incomplete and the interval of polar angles corresponds to a set of real numbers.

Remark 1. *A rose curve with the equation (4) for a >0, k = 1 in the polar coordinate system is associated with the cosine function y = acosx in the Cartesian plane. We therefore recall the following properties of the cosine function y = acosx in the Cartesian plane, which is a periodic function with a period 2π defined for all real numbers with values in the interval [−a,a]. The graph of the cosine function consists of an infinite number of parts of length 2π, which are called cycles and have the same shape. A cycle consists of two half-cycles, one positive and one negative, both of length π, therefore a half-cycle is bounded by an interval of length π. If we denote the centre of the interval by x_0, then a half-cycle is symmetric with respect to the line $x = x_0$ and has the vertex at (x_0,a) or $(x_0,-a)$, where the sign of a depends on the positivity or negativity of a half-cycle. The limits of each half-cycle are the points on the x-axis, which are also the limits of the corresponding interval. In particular, the interval $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$ of length 2π refers to a cycle whose positive half-cycle coincides with the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and a negative half-cycle is assigned to the interval $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$. The positive half-cycle is symmetrical with respect to the line x = 0 and has the vertex (maximum) at (0,a), and the negative half-cycle is symmetrical with respect to the line x = π and has the vertex (minimum) at (π,−a).*

Remark 2. *In general, the shape of a petal of a rose curve determined by (4) is formed by a half-cycle of the graph of the corresponding cosine function. The length of a cycle corresponds to the period* P *of the cosine function, which is given by*

$$\mathsf{P} = \frac{2\pi}{k}. \quad (9)$$

The length of each half-cycle is therefore equal to half the period P. According to Remark 1, we consider the positive half-cycle of the graph of the cosine function, which has the vertex at (0,a) and is assigned to the interval $\left[-\frac{\mathsf{P}}{4},\frac{\mathsf{P}}{4}\right]$ of length $\frac{\mathsf{P}}{2}$. The positive half-cycle forms the petal of a rose curve so that it begins and ends at the pole, has the peak at (a,0), is symmetric with respect to the polar axis and is bounded by the interval

$$\left[-\frac{\mathsf{P}}{4},\frac{\mathsf{P}}{4}\right] \quad . \quad (10)$$

of length $\frac{\mathsf{P}}{2}$, where P *is given by (9). The peak at (a,0) is the intersection of the polar axis with the circle of radius a whose center is the pole, whereby a rose curve is completely inscribed in this circle.*

Let n_p be the number of petals of a rose curve determined by (4). From the fact that every rose curve is inscribed in a circle of radius *a*, whose center is the pole, and consists of n_p equally shaped petals whose peaks lie on the circle, it follows that the distances between two peaks of successive petals are equal. Therefore, all polar angles between the peaks of successive petals are identical, which leads to the conclusion that the polar angle Φ between the peaks of the successive petals of a rose curve is given by

$$\Phi = \frac{2\pi}{n_p}, \quad (11)$$

where n_p denotes the number of petals of a rose curve. Here we use the fact that one full turn is equal to 2π.

Remark 3. *From the identity (11) it follows that the measure of the polar angle between the peaks of the successive petals of a rose curve depends on the number of petals of a rose curve, so that it decreases as the number of petals increases and the petals become narrower.*

Theorem 1. *Let us assume that a rose curve is determined by (4), where the angular frequency k of the cosine function is a positive integer. Then we distinguish the following cases.If k is even, then a rose curve consisting of 2k petals is complete for every continuous interval of length 2π.If k is odd, then a rose curve consisting of k petals is complete for every continuous interval of length π.The polar angle between the peaks of the successive petals of a rose curve is given by $\Phi = \frac{\pi}{k}$ if k is even, and $\Phi = \frac{2\pi}{k}$ by k is odd.*

Proof. According to Remark 2, in which the petal with the peak at (*a*,0) is considered, and from the fact that the graphs of all half-cycles have the same shape, it follows that all petals of a rose curve are equal, so that each petal begins and ends at the pole, has a peak on the circle of radius *a* whose center is the pole, and is symmetric with respect to the line passing through the pole and the peak of this petal.Assuming that a rose curve is determined by (4), where the angular frequency *k* of the cosine function is a positive integer, the period of the cosine function is less than or equal to 2π for each positive integer *k*. The cycles and also half-cycles can therefore be considered in every continuous interval of length 2π, which means that a rose curve determined by (4) for a positive integer angular frequency is complete for every continuous interval of length 2π. There are therefore *k* cycles and 2*k* *half-cycles*, of which *k* are positive and *k* negative half-cycles, which means that a rose curve consists of 2*k* petals.

However, it should be noted that the positive half-cycles coincide with the negative half-cycles in the case of an odd angular frequency, when the negative half-cycles can be obtained from the positive half-cycles by a horizontal shift of π units and vice versa. The petals formed from the positive half-cycles are identical to the petals formed from the negative half-cycles. It is therefore sufficient to consider only the positive half-cycles or only the negative half-cycles, which means that a rose curve determined by (4) for an odd angular frequency consists of *k* petals and is complete for each continuous interval of length π. On the other hand, a rose curve determined by (4) for an even angular frequency consists of 2*k* petals and is complete for each continuous interval of length 2π. From (11) it follows that $\Phi = \frac{\pi}{k}$ if *k* is even and $\Phi = \frac{2\pi}{k}$ if *k* is odd.

Considering the identity (9), the length of each half-cycle of the graph of the cosine function with a positive integer angular frequency is equal to $\frac{\pi}{k}$. Therefore, the petal, which has the peak at $(a,0)$ on the polar axis, is bounded by the interval

$$\left[-\frac{\pi}{2k}, \frac{\pi}{2k}\right] \qquad (12)$$

of length $\frac{\pi}{k}$. Based on the first two statements of Theorem 1, we can assume that a rose curve determined by (4) for an even angular frequency is complete for the following continuous interval

$$\left[-\frac{\pi}{2k}, \frac{(4k-1)\pi}{2k}\right] \qquad (13)$$

of length 2π. Similarly, a rose curve determined by (4) for an odd angular frequency is complete for the continuous interval

$$\left[-\frac{\pi}{2k}, \frac{(2k-1)\pi}{2k}\right]. \qquad (14)$$

of length π. It is obvious that these intervals (13) and (14) are not the only intervals for which the corresponding rose curve is complete. They are one of an infinite number of intervals for which a rose curve is complete. They were chosen because they contain the interval (12) that bounds the petal with the peak at $(a,0)$ on the polar axis.

From the comparison of the third statement of Theorem 1 with the period of the cosine function given by (9), it follows that the polar angle between the peaks of the successive petals of a rose curve is equal to the period of the cosine function if its angular frequency is odd. If the angular frequency of the cosine function is even, then the polar angle between the peaks of the successive petals of a rose curve is equal to half the period of the cosine function.

Example 4. *Here we look at three rose curves determined by* (4)*, where the amplitude of the corresponding cosine function is equal to one, which means that these rose curves are inscribed in a circle of radius one whose center is the pole. Figure 5 shows in particular the rose curve* $r = \cos\phi$ *on the left-hand side, which is connected to the cosine function whose amplitude and angular frequency are equal to one. Looking at the corresponding cosine function in the Cartesian plane, see Remark 1, it is easy to prove that the positive half-cycle coincides with the negative half-cycle and forms a petal bounded by an interval of length* π*. The rose curve* $r = \cos\phi$ *therefore consists of a petal, which is actually a circle of radius one whose center is the point* $(\frac{1}{2},0)$.

In addition, with regard to Theorem 1, we also consider the two rose curves $r = \cos 2\phi$ *and* $r = \cos 3\phi$*, which are shown in the center and on the right in Figure5. The rose curve* $r = \cos 3\phi$*, also called trifolium, consists of 3 petals and is complete for each continuous interval of length* π*, see Figure 5 on the right. The peaks of the corresponding petals of this rose curve are the points* $\left(1, i \cdot \frac{2\pi}{3}\right)$ *for* $i = 0,1,2$*, which lie on the circle of radius one whose center is the pole. Therefore, the polar angle between the peaks of the successive petals is given by* $\Phi = \frac{2\pi}{3}$*, see the second and third statements of Theorem 1. If we take into account that the period of the corresponding cosine function is given by* $P = \frac{2\pi}{3}$*, we obtain from the application of* (10) *or equivalently* (12)*, that the petal with the peak at* (1,0) *is bounded by the interval* $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ *of length* $\frac{\pi}{3}$*. We can therefore assume that the rose curve* $r = \cos 3\phi$ *is complete on the interval* $\left[-\frac{\pi}{6}, \frac{5\pi}{6}\right]$ *of length* π*, see* (14)*.*

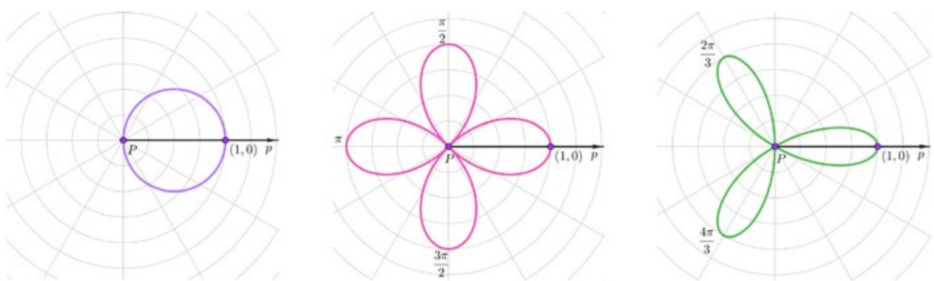

Figure 5: Rose Curves $r = \cos\phi$, $r = \cos 2\phi$ and $r = \cos 3\phi$.

The first and third statements of Theorem 1 can be illustrated using the rose curve $r = \cos 2\phi$*, also called quadrifolium, see Figure 5 in the middle. Since the angular frequency of the cosine function is even, this rose curve consists of* 4 *petals and is complete for each continuous interval of length* 2π*. Therefore, the peaks of the corresponding petals of this rose curve are the points* $\left(1, i \cdot \frac{\pi}{2}\right)$ *for* $i = 0, 1, 2, 3$*, which lie on the circle of radius one whose center is the pole. The polar angle between the peaks of the successive petals is therefore given by* $\Phi = \frac{\pi}{2}$*. If we consider the period* $P = \pi$ *of the corresponding cosine function, the petal with the peak at (1,0) is bounded by the interval* $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ *of length* $\frac{\pi}{2}$*, so it can be assumed that this rose curve is complete on the interval* $\left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$ *of length* 2π*, see (13). If we look at Figure 5, we see that each petal of the rose curve is symmetrical about the line through the pole and its peak.*

In accordance with Theorem 1 and Remark 3, we show in the following two figures that as the number of petals in a rose curve increases, the polar angle between the peaks of successive petals decreases, making the petals narrower. In addition, the rose curves are shown in Figure 6 with respect to odd angular frequencies and in Figure 7 with respect to even angular frequencies of the corresponding cosine functions. We have also assumed that the amplitude of the corresponding cosine functions is equal to one, which means that these rose curves are inscribed in a circle of radius one whose center is the pole.

The two rose curves in Figure 6 are complete for each continuous interval of length π. The left side of Figure 6 shows that the rose curve $r = \cos 7\phi$ consists of 7 petals, therefore $\Phi = \frac{2\pi}{7}$ is the polar angle between the peaks of the successive petals, which is also equal to the period of the cosine function, see (9). The peaks of the corresponding petals of the rose curve are $\left(1, i \cdot \frac{2\pi}{7}\right)$ for $i = 0, 1, 2, \ldots 6$. The petal with the peak at (1,0) on the polar axis is bounded by the interval $\left[-\frac{\pi}{14}, \frac{\pi}{14}\right]$of length $\frac{\pi}{7}$ (see (12)), so the interval $\left[-\frac{\pi}{14}, \frac{13\pi}{14}\right]$is one of infinitely many intervals for which this rose curve is complete, see (14).

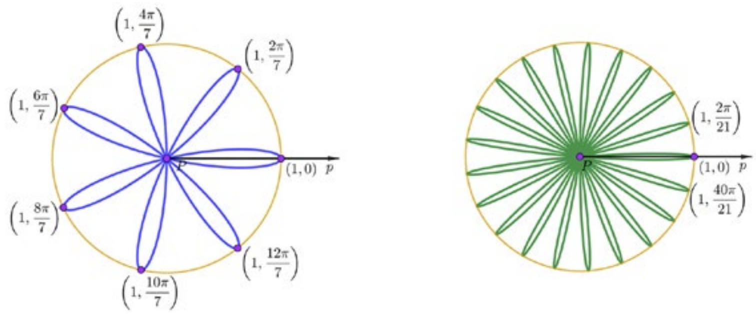

Figure 6: Rose Curves r = cos7ϕ and r = cos21ϕ.

Similarly, the right side of Figure 6 shows that the rose curve $r = \cos 21\phi$ consists of 21 petals, so that in this case the polar angle between the peaks of the successive petals is given by $\Phi = \frac{2\pi}{21}$, which means that the peaks of the corresponding petals are the points $\left(1, i \cdot \frac{2\pi}{21}\right)$ for i = 0,1,2,...,20. The petal with the peak at (1,0) on the polar axis is bounded by the interval $\left[-\frac{\pi}{42}, \frac{\pi}{42}\right]$of length $\frac{\pi}{21}$, therefore the interval $\left[-\frac{\pi}{42}, \frac{41\pi}{42}\right]$ is one of an infinite number of intervals for which this rose curve is complete.

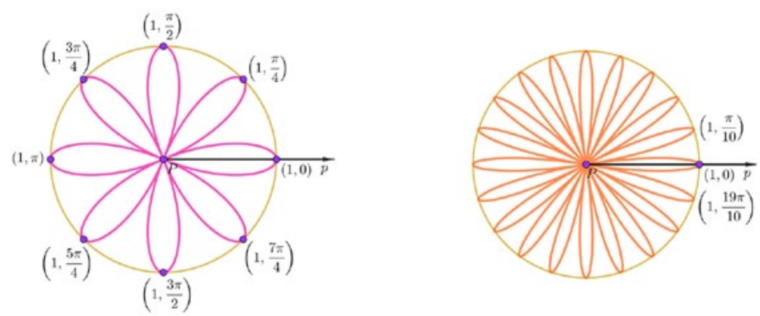

Figure 7: Rose Curves r = cos4ϕ and r = cos10ϕ.

Considering the even angular frequency, Figure 7 shows the rose curve $r = \cos 4\phi$ on the left and the rose curve $r = \cos 10\phi$ on the right, both of which are complete for each continuous interval of length 2π. The rose curve $r = \cos 4\phi$ consists of 8 petals, so $\Phi = \frac{\pi}{4}$ is the polar angle between the peaks of the successive petals, which is equal to half the period of the corresponding cosine function. If we apply (12), we obtain that the petal with the peak at (1,0) is bounded by the interval $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$belongs to the interval $\left[-\frac{\pi}{8}, \frac{15\pi}{8}\right]$ for which this rose curve is complete, see (13). Taking into account the determined polar angle $\Phi =\frac{\pi}{4}$ between the peaks of the successive petals, we obtain that $\left(1, i \cdot \frac{\pi}{4}\right)$ for *i = 0,1,2,...,7* are the peaks of the corresponding petals of this rose curve. Similarly, the rose curve $r = \cos 10\phi$ consists of 20 petals. The petal with the peak at (1,0) is bounded by the interval $\left[-\frac{\pi}{20}, \frac{\pi}{20}\right]$ belongs to the interval $\left[-\frac{\pi}{20}, \frac{39\pi}{20}\right]$ for which this rose curve is complete. The polar angle between the peaks of the successive petals is given by $\Phi = \frac{\pi}{10}$, which means that the peaks of the corresponding petals of this rose curve are the points $\left(1, i \cdot \frac{\pi}{10}\right)$ for i = 0, 1, 2,..., 19.

In the following, we assume that the angular frequency of the cosine function in equation (4) is a positive rational number whose numerator m and denominator n are positive integers, so that they are relatively prime. In this way, we consider that k has the form of an irreducible fraction given by $k = \frac{m}{n}$, so it follows from (9) that the period of the cosine function is given by

$$\mathsf{P} = \frac{2n\pi}{m}. \tag{15}$$

Theorem 2. *Suppose that a rose curve is determined by (4), where the angular frequency k of the cosine function is a positive rational number that has the form of an irreducible fraction given by* $k = \frac{m}{n}$ *with m,n*>0. *Then we have the following cases.If m is odd and n is even, or vice versa, then a rose curve consisting of* $2m$ *petals is complete for every continuous interval of length* $2n\pi$. *If both m and n are odd, then a rose curve consisting of m petals is complete for every continuous interval of length nπ. The polar angle between the peaks of the successive petals of a rose curve is given by* $\Phi = \frac{\pi}{m}$ *in the first case and by* $\Phi = \frac{2\pi}{m}$ *in the second case.*

Proof. From the assumption of Theorem 2 and the observation in Remark 2, it follows that the length of each half-cycle is equal to $\frac{n\pi}{m}$. The positive half-cycle of the graph of the cosine function, which has the vertex at (0,a), therefore forms the petal with the peak at (a,0) on the polar axis, which is bounded by the interval

$$\left[-\frac{n\pi}{2m}, \frac{n\pi}{2m}\right] \quad (16)$$

of length $\frac{n\pi}{m}$. Note that (16) follows from (10) for the period (15) of the cosine function under consideration. The period (15) of the cosine function leads to the conclusion that a rose curve determined by (4), where the angular frequency $k = \frac{m}{n}$ of the cosine function is a positive rational number, is complete for any continuous interval of length $2n\pi$ or $n\pi$, which depends on the even and odd numerators and denominators of the angular frequency. Similar to the case of an integer angular frequency, we distinguish two cases here: the first, when one of the numerators and denominators of the angular frequency is even and the other odd, and the second, when both are odd. In the first case, all half-cycles are different, but in the second case, the positive half-cycles coincide with the negative half-cycles, so that a rose curve is complete for each continuous interval of length $2n\pi$ in the first case and length $n\pi$ in the second case.

With respect to an interval of length $2n\pi$, there are m cycles and $2m$ half-cycles, of which all m positive and m negative half-cycles are different if one of m and n is even and the other is odd. In this case, the rose curve therefore consists of $2m$ petals, since the number of half-cycles is equal to the number of petals of the rose curve. If, on the other hand, both m and n are odd, the positive half-cycles coincide with the negative half-cycles, so that the rose curve consists of m petals. Finally, it follows from (11) that the polar angle between the peaks of the successive petals of a rose curve is given by $\Phi = \frac{\pi}{m}$ if one of m and n is even and the other is odd, and by $\Phi = \frac{2\pi}{m}$ if both m and n are odd.

In the special case, if the denominator of the angular frequency $k = \frac{m}{n}$ is equal to one, then $k = m$, so that Theorem 1 follows from Theorem 2. Using Theorem 2, we obtain that each peak of the corresponding petal of a rose curve determined by (4), where the angular frequency k of the cosine function is a positive rational number, belongs to the set of points $\left(a, i \cdot \frac{\pi}{m}\right)$ for $i = 0,1,2,...,2m - 1$ in the first case and the points $\left(a, i \cdot \frac{2\pi}{m}\right)$ for $i = 0, 1, 2,..., m-1$ in the second case, which lie on a circle of radius a whose center is the pole.

If we consider the petal with the peak at $(a,0)$ on the polar axis bounded by the interval (16), a rose curve determined by (4), where the angular frequency $k = \frac{m}{n}$ of the cosine function is a positive rational number, is complete for the interval

$$\left[-\frac{n\pi}{2m}, \frac{n(4m-1)\pi}{2m}\right], \quad (17)$$

if one of m and n is even and the other is odd, and for the interval

$$\left[-\frac{n\pi}{2m}, \frac{n(2m-1)\pi}{2m}\right], \quad (18)$$

if both m and n are odd, where the length of the interval (17) is equal to $2n\pi$ and the length of the interval (18) is equal to $n\pi$. The intervals (17) and (18), which contain the interval (16), are one of an infinite number of intervals for which a corresponding rose curve is complete.

We now briefly explain the shape of the petals of a rose curve, which is determined by (4), where the cosine function has a positive rational angular frequency $k = \frac{m}{n}$ and m and n are relatively prime. Since all petals of the rose curve are identical, it is sufficient to consider only the shape of one petal.Depending on the period P of the cosine function with positive rational angular frequency, the shape of a petal can be either a single closed loop or a petal forming multiple loops, where each petal of a rose curve begins and ends at the pole, has a peak on the circle of radius a whose center is the pole, and is symmetrical with respect to the line passing through the pole and the peak of a petal.

From the fact that the shape of a petal of a rose curve is formed by a half-cycle whose length is equal to the half period P given by (15), the shape of a petal is a single closed loop if the half period of the cosine function is less than or equal to 2π or if the period P of the cosine function is less than or equal to 4π. In addition, a petal forms two loops if the period P of the cosine function fulfils the condition $4\pi < P \leq 8\pi$. Similarly, a petal forms three loops if the period P of the cosine function fulfils the condition $8\pi < P \leq 12\pi$. In general, a petal forms ξ loops if the period P of the cosine function fulfils the condition

$$4(\xi - 1)\pi < P \leq 4\xi\pi, \quad (19)$$

where ξ is a positive integer. The shape of a petal of a rose curve is also a single closed loop if a rose curve is determined by (4), where the cosine function has a positive integer angular frequency. Note that in this case the period of the cosine function is always less than or equal to 2π, which is in agreement with (19) for $\xi = 1$.

The petals of a rose curve relating to a positive rational angular frequency overlap, in contrast to the petals of a rose curve relating to a positive integer angular frequency, which do not overlap. In the case of a rational angular frequency, similar to the case of an integer angular frequency, the distance between successive petals, i.e. the polar angle between the peaks of the successive petals of a rose curve, decreases as the number of petals increases. We distinguish between the following two cases.

1. If the rational angular frequency is less than one and decreases towards zero, then the petals become wider. In this case, it is possible that a petal of a rose curve is a single closed loop or forms multiple loops.
2. If the rational angular frequency is greater than one, then the petal becomes narrower and increases. In this case, each petal is a single closed loop, as in the case of an integer angular frequency.

With reference to Theorem 2, in the following three examples we examine several rose curves, each of which is associated with the corresponding rational angular frequency of the cosine function. In particular, in Example 5 and Example 6 we consider rational angular frequencies less than one and in Example 7 rational angular frequencies greater than one, assuming that the amplitude of the corresponding cosine functions is equal to one.

Example 5. *We begin with Figure 8 and Figure 9 which show rose* *curves, each of which is associated with the corresponding rational angular frequency of the cosine function whose numerator is one and whose denominator is odd in Figure 8 and even in Figure 9. Figure 10 shows a petal of the corresponding rose curve in Figure 9.*

Figure 8 shows in particular, from left to right, the rose curves $r=\cos\frac{1}{3}\phi$, $r=\cos\frac{1}{5}\phi$, $r=\cos\frac{1}{7}\phi$, *each consisting of a petal forming several loops. According to the second statement of Theorem 2, these rose curves are complete for each continuous interval of length* 3π, 5π, 7π *in the order given. Since the period P of the corresponding cosine function is equal to* 6π, 10π, 14π *in the given order of the rose curves, the petal (i.e. the rose curve) forms* 2, 3, 4 *loops in the given order, see (19) and also Figure 8. Furthermore, the points* A, B_1, B_2, C_1, C_2, C_3 *are the intersections of the corresponding rose curve with itself. More precisely, the rose curve* $r=\cos\frac{1}{3}\phi$

intersects at the point $A=\left(\cos\frac{\pi}{3},\pi\right)=\left(\frac{1}{2},\pi\right)$, *gives the same point for the polar angle* $\phi=-\pi$, *where* $\left(\cos\left(-\frac{\pi}{3}\right),-\pi\right)=\left(\cos\frac{\pi}{3},-\pi\right)$, *see* (7). *Similarly, the intersections of the rose curve* $r=\cos\frac{1}{5}\phi$ *with itself are* $B_1=\left(\cos\frac{\pi}{5},\pi\right)=\left(\cos\frac{\pi}{5},-\pi\right)$ *and* $B_2=\left(\cos\frac{2\pi}{5},2\pi\right)=\left(\cos\frac{2\pi}{5},-2\pi\right)$ *and the intersection points of the rose curve* $r=\cos\frac{1}{7}\phi$ *with itself are* $C_1=\left(\cos\frac{\pi}{7},\pi\right)=\left(\cos\frac{\pi}{7},-\pi\right)$, $C_2=\left(\cos\frac{2\pi}{7},2\pi\right)=\left(\cos\frac{2\pi}{7},-2\pi\right)$ *and* $C_3=\left(\cos\frac{3\pi}{7},3\pi\right)=\left(\cos\frac{3\pi}{7},-3\pi\right)$.

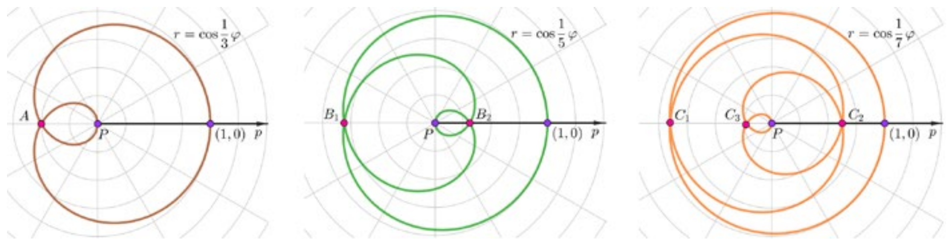

Figure 8: Rose Curves $r=\cos\frac{1}{3}\phi$, $r=\cos\frac{1}{5}\phi$ and $r=\cos\frac{1}{7}\phi$ Consist of One Petal Forming Several Loops.

Let us now consider the rose curves that refer to the even denominators of the angular frequencies whose numerator is one. These rose curves then consist of two petals that are symmetrical with respect to the line $\phi=\frac{\pi}{2}$. *In particular, Figure 9 shows the rose curves* $r=\cos\frac{1}{2}\phi$, $r=\cos\frac{1}{4}\phi$, $r=\cos\frac{1}{6}\phi$ *from left to right.*

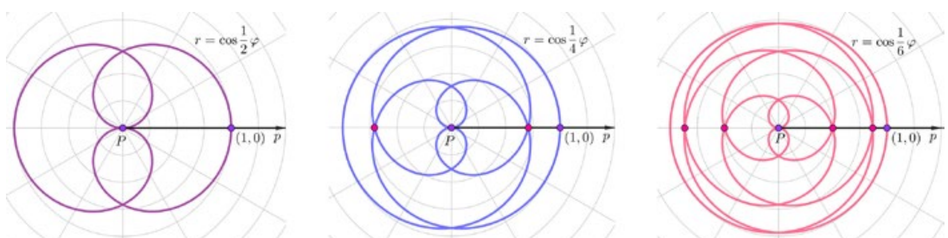

Figure 9: Rose Curves $r=\cos\frac{1}{2}\phi$, $r=\cos\frac{1}{4}\phi$, $r=\cos\frac{1}{6}\phi$ consist of two petals.

If we use the first statement of Theorem 2 here, we obtain that these rose curves are complete for each continuous interval of length 4π, 8π, 12π *in the given order of rose curves in the observation. To better explain the number of loops of a petal of these rose curves, consider the petal with the peak at* (1,0) *of each of these rose curves, as shown in Figure 10, where the intersection points of the petal with itself are also indicated.*

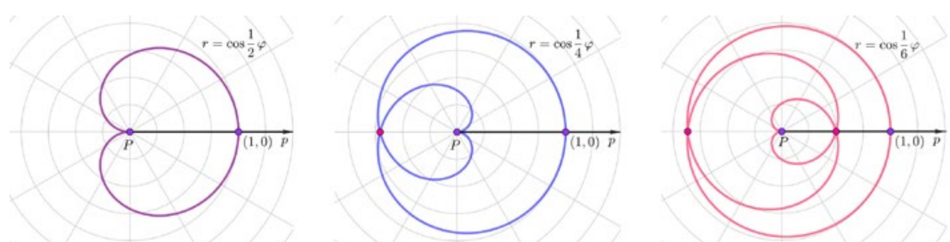

Figure 10: The Petal of The Rose Curves $r=\cos\frac{1}{2}\phi$, $r=\cos\frac{1}{4}\phi$ and $r=\cos\frac{1}{6}\phi$.

According to the identity (19) and the period P = 4π of the cosine function with respect to the rose curve $r=\cos\frac{1}{2}\phi$, *we obtain that each of the two petals is a single closed loop, which means that the petal does not intersect itself, see Figure 10 left. With respect to the rose curve* $r=\cos\frac{1}{4}\phi$, *where the period of the corresponding cosine function is equal to P = 8π, it follows from (19) that each of the two petals forms* 2 *loops and intersects itself at the point*

$$\left(\cos\frac{\pi}{4},\pi\right)=\left(\frac{\sqrt{2}}{2},\pi\right),\ where\ \left(\cos\frac{\pi}{4},\pi\right)=\left(\cos\frac{\pi}{4},-\pi\right)$$

see Figure 10 in the middle. Finally, we obtain in a similar way that each of the two petals of the rose curve $r=\cos\frac{1}{6}\phi$ forms 3 loops, since the period of the corresponding cosine function is equal to 12π. The intersection points of this rose curve with itself are $\left(\cos\frac{\pi}{6},\pi\right)=\left(\frac{\sqrt{3}}{2},\pi\right)$ *and* $\left(\cos\frac{\pi}{3},2\pi\right)=\left(\frac{1}{2},2\pi\right)$, Here we have considered $\left(\cos\frac{\pi}{6},\pi\right)=\left(\cos\frac{\pi}{6},-\pi\right)$ *and* $\left(\cos\frac{\pi}{3},2\pi\right)=\left(\cos\frac{\pi}{3},-2\pi\right)$, see Figure 10 on the right. If we compare each of the rose curves in Figure 9 with the corresponding petal in Figure 10, we can see that the two petals of a rose curve overlap.

Example 6. *We now examine the rose curves that refer to rational angular frequencies less than one whose numerators are different from one and where one of the numerators or denominators is even and the other odd, as well as the case where both are odd. In particular, Figure 11 shows the rose curve* $r=\cos\frac{3}{4}\phi$ *on the left and the rose curve* $r=\cos\frac{3}{5}\phi$ *on the right. By applying (19) we find that each of the petals of the two rose curves is a single closed loop, since the period of the cosine function is given by* $\frac{8\pi}{3}\le 4\pi$ *and* $\frac{10\pi}{3}\le 4\pi$ *respectively. According to Theorem 2, we find that the rose curve* $r=\cos\frac{3}{4}\phi$ *is complete for every continuous interval of length 8π and consists of 6 petals, of which the petal with the peak at (1,0) is bounded by the interval* $\left[-\frac{2\pi}{3},\frac{2\pi}{3}\right]$ *, which belongs to the interval* $\left[-\frac{2\pi}{3},\frac{22\pi}{3}\right]$ *, see (16) and (17). With (11) for* $n_p=6$ *we obtain that the polar angle between the peaks of the successive petals is given by* $\Phi=\frac{\pi}{3}$*. Therefore, the peaks of the corresponding petals are points* $\left(1,i\cdot\frac{\pi}{3}\right)$ *for i = 0,1,2,...,5*

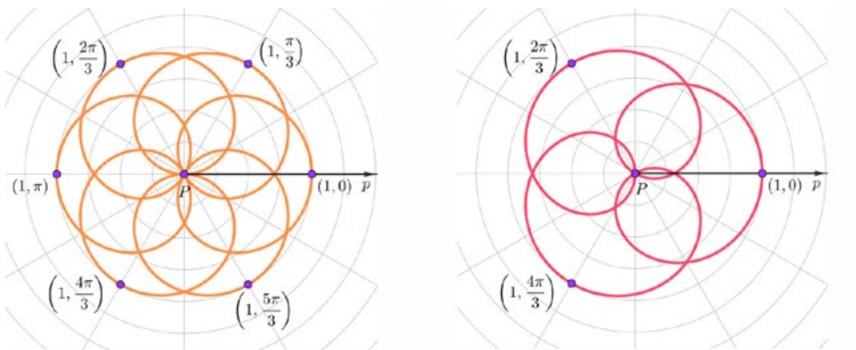

Figure 11: Rose Curves $r=\cos\frac{3}{4}\phi$ and $r=\cos\frac{3}{5}\phi$.

On the other hand, since both the numerator and the denominator of the angular frequency of the cosine function with respect to the rose curve $r=\cos\frac{3}{5}\phi$ *are odd, this rose curve is complete for every continuous interval of length* 5π *and consists of* 3 *petals, of which the petal with the peak at (*1,0*) is bounded by the interval* $\left[-\frac{5\pi}{6},\frac{5\pi}{6}\right]$ *, which belongs to the interval* $\left[-\frac{5\pi}{6},\frac{25\pi}{6}\right]$*, see (*16*) and (*18 *). The polar angle between the peaks of the successive petals is given by* $\Phi=\frac{2\pi}{3}$ *and* $\left(1,i\cdot\frac{2\pi}{3}\right)$, $i=0,1,2$ *are the peaks of the corresponding petals.*

We give here two more rose curves that refer to rational angular frequencies less than one and to a period greater than 4π*, which results in each of the petals forming multiple loops, as shown in Figure 12 and Figure 13.*

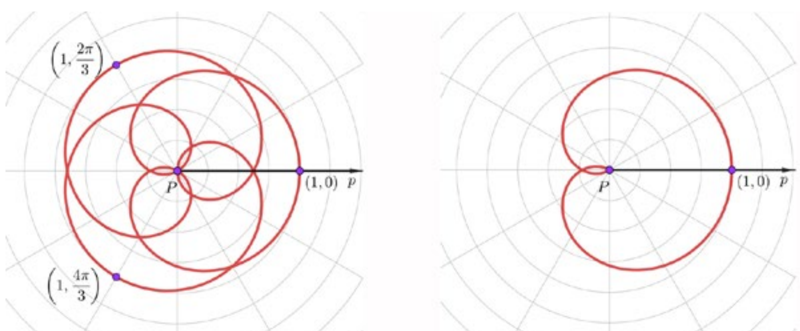

Figure 12: The Rose Curve $r=\cos\frac{3}{7}\phi$ and its petal forming 2 loops.

In particular, Figure 12 shows on the left the rose curve $r=\cos\frac{3}{7}\phi$ *, which consists of* 3 *petals and is complete for each continuous interval of length* 7π*. Since the period* P *of the cosine function is equal to* $\frac{14\pi}{3}$*, which is greater than 4π and less than or equal to 8π, it follows from (*19*) that each of the three petals forms* 2 *loops, as Figure 12 on the right shows.*

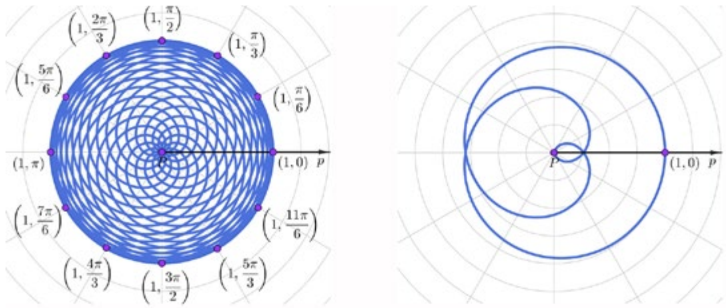

Figure 13: The rose curve $r = \cos\frac{6}{29}\phi$ and its petal forming 3 loops.

Similarly, Figure 13 on the left shows the rose curve $r = \cos\frac{6}{29}\phi$ *, which is complete for each continuous interval of length 58π and consists of 12 petals whose corresponding peaks are the points* $\left(1, i \cdot \frac{\pi}{6}\right)$ *for i = 0,1,2,...,11. The period of the cosine function with respect to this rose curve is given by* $\mathsf{P} = \frac{29\pi}{3}$*, so that from* $8\pi < \mathsf{P} \leq 12\pi$ *it follows that each of the petals of this rose curve forms 3 loops, as Figure 13 on the right shows.*

If we compare the treated rose curves in Figure 11, Figure 12, and Figure 13, we can see that each of the first three is associated with the rational angular frequency whose numerator is equal to 3. This may mean that each consists of 3 petals, but only the rose curves $r = \cos\frac{3}{5}\phi$ and $r = \cos\frac{3}{7}\phi$ consist of 3 petals, because the numerator and denominator of the angular frequency are odd, in contrast to the rose curve $r = \cos\frac{3}{4}\phi$, which consists of 6 petals because one of the numerators and denominators of the angular frequency is odd and the other is even. Recall that the rose curve $r = \cos 3\phi$ shown in Figure 5 on the right and each rose curve $r = a\cos 3\phi$, $a>0$ also consists of 3 petals. The common property of the rose curves $r = \cos\frac{3}{5}\phi$, $r = \cos\frac{3}{7}\phi$ and $r = \cos 3\phi$ is that the polar coordinates of the peaks of their corresponding petals coincide and they therefore have the same polar angle between the peaks of the successive petals, which is given by $\Phi = \frac{2\pi}{3}$. On the other hand, they differ in the length of the interval for which the corresponding rose curve is complete, but also in the number of loops of the petal, which depends on the period of the associated cosine function. As mentioned, each petal of the rose curves $r = \cos\frac{3}{5}\phi$ and $r = a\cos 3\phi$ is a single closed loop, but each petal of the rose curve $r = \cos\frac{3}{7}\phi$ forms 2 loops.

Example 7. *We now examine two rose curves that refer to rational angular frequencies greater than one, where the amplitudes of the two cosine functions are also equal to one. In the first case, shown on the left in Figure 14, we assume that the numerator of the angular frequency of the cosine function is even and the denominator is odd. In the second case, shown on the right in Figure 14, we assume that the numerator and denominator of the angular frequency of the cosine function are odd. In particular, Figure 14 shows the rose curve* $r = \cos\frac{4}{3}\phi$ *on the left and the rose curve* $r = \cos\frac{7}{3}\phi$ *on the right. The common characteristics of these rose curves, and also of all rose curves associated with the cosine function with rational angular frequencies greater than one, are that each of the petals is a single closed loop and that the petals overlap, unlike the rose curves associated with the cosine function with integer angular frequencies, whose petals do not overlap. Similar to the integer angular frequency, the polar angle between the peaks of successive petals, decreases as the number of petals increases. If the rational angular frequency k is greater than one, then the petal becomes narrower as k increases from one and wider as k decreases towards one.*

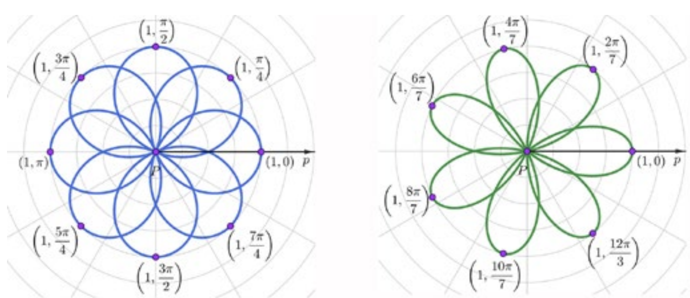

Figure 14: Rose Curves $r = \cos\frac{4}{3}\phi$ and $r = \cos\frac{7}{3}\phi$

By applying Theorem 2, it is easy to verify that the rose curve $r = \cos\frac{4}{3}\phi$ *is complete for each continuous interval of length* 6π *and consists of 8 petals whose corresponding peaks are the points* $\left(1, i \cdot \frac{\pi}{4}\right)$ *for* $i = 0, 1, 2, \ldots, 7$ *, as shown in Figure 14 left. The petal with the peak at (*1,0*) is bounded by the interval* $\left[-\frac{3\pi}{8}, \frac{3\pi}{8}\right]$, *which belongs to the interval* $\left[-\frac{3\pi}{8}, \frac{45\pi}{8}\right]$*. Similarly, one can verify that the rose curve* $r = \cos\frac{7}{3}\phi$ *is complete for each continuous interval of length 3π and consists of 7 petals whose corresponding peaks are the points* $\left(1, i \cdot \frac{2\pi}{7}\right)$ *for i =* 0,1,2,...,6*, see Figure 14 right.*

*The petal with the peak at (*1,0*) is bounded by the interval* $\left[-\frac{3\pi}{14}, \frac{3\pi}{14}\right]$*, which belongs to the interval* $\left[-\frac{3\pi}{14}, \frac{39\pi}{14}\right]$.

In the following, we finally examine the rose curves with regard to irrational angular frequencies.

Theorem 3. *If the angular frequency k of the cosine function is a positive irrational number, then a rose curve consists of an infinite number of petals. Proof.* Assuming that a rose curve is determined by (4), where the angular frequency k of the cosine function is a positive irrational number, the period of the cosine function can have a rational or irrational value. In particular, the period P of the cosine function has a rational value if an irrational angular frequency is given in the form of $k = \lambda \bullet \pi$, where λ is a rational number, otherwise the period P has an irrational value. In both cases, however, the period P cannot be represented as the product of a rational number and the number π, in contrast to the case of a rational angular frequency, where the period P is given as the product of a rational number $\frac{2n}{m}$ and the number π, see (15). Since a cycle is a part of the graph of the cosine function whose length corresponds to a period of the cosine function, there are no cycles that coincide. Consequently, all cycles are accounted for, which means that there are no identical petals, resulting in a rose curve consisting of an infinite number of petals.

It follows from Theorem 3 that a rose curve determined by (4), where the angular frequency of the cosine function is a positive irrational number, is incomplete in the set of real numbers. On the other hand, it follows from the fact that a rose curve consists of an infinite number of petals that the polar angle between the peaks of the successive petals of a rose curve is not unique. In fact, it decreases as the number of petals increases.

Theorem 4. *Let us assume that a rose curve is determined by (4), where the angular frequency of the cosine function is a positive irrational number. Assuming that P denotes the period of the cosine function, the following statements are fulfilled.The petal with the peak at* $\left(a, \frac{l \cdot \mathsf{P}}{2}\right)$*, is bounded by the interval*

$$\left[\frac{(2l-1)\cdot \mathsf{P}}{4}, \frac{(2l+1)\cdot \mathsf{P}}{4}\right] \qquad (20)$$

of length $\frac{\mathsf{P}}{2}$ *, where l is any integer. The j −i+1 petals of a rose curve are bounded by the interval*

$$\left[\frac{(2i-1)\cdot \mathsf{P}}{4}, \frac{(2j+1)\cdot \mathsf{P}}{4}\right] \qquad (21)$$

of length $\frac{(j-i+1)\cdot \mathsf{P}}{2}$*, where i and j are arbitrary integers such that* $i \leq j$.

Proof. Since the shape of each petal is the same and each petal is bounded by the corresponding interval of length $\frac{\mathsf{P}}{2}$, we first consider the petal with the peak at (a,0), which is bounded by the interval $\left[-\frac{\mathsf{P}}{4}, \frac{\mathsf{P}}{4}\right]$, whose centre is $\phi = 0$, see (10). The centre of an interval (of polar angles) is connected to the peak of a petal that is bounded by this interval. So if we consider the interval $\left[\frac{(2l-1)\cdot \mathsf{P}}{4}, \frac{(2l+1)\cdot \mathsf{P}}{4}\right]$ for any integer l whose centre is $\phi = \frac{l \cdot \mathsf{P}}{2}$, then we obtain that the petal with the peak at $\left(a, \frac{l \cdot \mathsf{P}}{2}\right)$, is bounded by the interval (20) whose centre is $\phi = \frac{l \cdot \mathsf{P}}{2}$.

In accordance with (20) for every integer l, we obtain that the petal with the peak at $\left(a, \frac{(l+1)\cdot \mathsf{P}}{2}\right)$ is bounded by the interval $\left[\frac{(2l+1)\cdot \mathsf{P}}{4}, \frac{(2l+3)\cdot \mathsf{P}}{4}\right]$, whose centre is $\phi = \frac{(l+1)\cdot \mathsf{P}}{2}$. If we now consider the interval (20) together with the interval obtained, we find that the two petals, one of which has the peak at $\left(a, \frac{l \cdot \mathsf{P}}{2}\right)$, and the other at $\left(a, \frac{(l+1)\cdot \mathsf{P}}{2}\right)$, are bounded by the interval

$$\left[\frac{(2l-1)\cdot \mathsf{P}}{4}, \frac{(2l+3)\cdot \mathsf{P}}{4}\right], \qquad (22)$$

of length P. Considering that $2l + 3 = 2(l + 1) + 1$ and introducing the abbreviations $i = l$ and $j = l+1$, we obtain that the $j - i + 1 = 2$ petals are bounded by the interval (22), which can be written as $\left[\frac{(2i-1)\cdot \mathsf{P}}{4}, \frac{(2j+1)\cdot \mathsf{P}}{4}\right]$, compare with (21). Therefore, the $j - i + 1$ petals of a rose curve are generally bounded by the interval (21) of length $\frac{(j-i+1)\cdot \mathsf{P}}{2}$ for any integers i and j such that $i \leq j$.

Although a rose curve is incomplete with respect to the irrational angular frequency of the cosine function and consists of an infinite number of petals, in general any finite number of its petals is bounded by an interval that is a subset of the set of real numbers. As the length of the interval (21) increases, the number of petals increases and thus the polar angle between the peaks of the successive petals decreases. It is obvious that all the peaks of the corresponding petals of a rose curve with respect to the irrational angular frequency of the cosine function associated with (4) lie on the circle of radius a whose center is the pole. In the special case for $i = j = l$, the interval (20) follows from (21). We note that the interval (21) of polar angles of length $\frac{(j-i+1)\cdot \mathsf{P}}{2}$ with $i \leq j$ bounding $j - i + 1$ petals of a rose curve with respect to an irrational angular frequency can be considered as a generalization of the above intervals of polar angles with respect to the integer or the rational angular frequency of the corresponding cosine function. In particular, if we assume that $i = j = 0$, then from (21) follows the interval (10) bounding the petal with the peak on the polar axis. It is also easy to check that in the case of an integer angular frequency the intervals (12), (13) and (14) can be determined from (21) and also in the case of a rational angular frequency the intervals (16), (17) and (18) can be determined from (21).

Example 8. *Let us consider the rose curve* $r = \cos(2\pi\phi)$*. Since the angular frequency of the cosine function is given by* $k = 2\pi$*, the period of the cosine function is equal to one, see (9). Using (20) with* $l = 0$ *and* P = 1 *(or equivalently (10) with* P = 1*), we obtain that the petal with the peak at (a,0) on the polar axis is bounded by the interval* $\left[-\frac{1}{4}, \frac{1}{4}\right]$ *of length* $\frac{1}{2}$*. In addition, the petal with the peak at* $\left(a, \frac{l}{2}\right)$ *is bounded by* the interval $\left[\frac{2l-1}{4}, \frac{2l+1}{4}\right]$ *, where l is an integer. In particular, if we assume that* $l = -3$*, then according to (20) the petal with the peak at* $\left(1, -\frac{3}{2}\right)$ *is bounded by the interval* $\left[-\frac{7}{4}, -\frac{5}{4}\right]$*. Similarly, for* $l = -2,-1,0,1,2,3$ *we note that the sequence of the following six petals with the corresponding peak at* $(1,-1)$, $\left(1,-\frac{1}{2}\right)$, $(1,0)$, $\left(1,\frac{1}{2}\right)$, $(1,1)$, $\left(1,\frac{3}{2}\right)$ *are bounded by the corresponding interval in the given sequence* $\left[-\frac{5}{4}, -\frac{3}{4}\right]$, $\left[-\frac{3}{4}, -\frac{1}{4}\right]$, $\left[-\frac{1}{4}, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{3}{4}\right]$, $\left[\frac{3}{4}, \frac{5}{4}\right]$, $\left[\frac{5}{4}, \frac{7}{4}\right]$*. In fact, these intervals are contained in the interval* $\left[-\frac{7}{4}, \frac{7}{4}\right]$*,which corresponds to* *the interval (21) for $i = -3$ and $j = 3$, as shown in Figure 15, top left, where the polar coordinates of the peaks of the corresponding petals are given.*

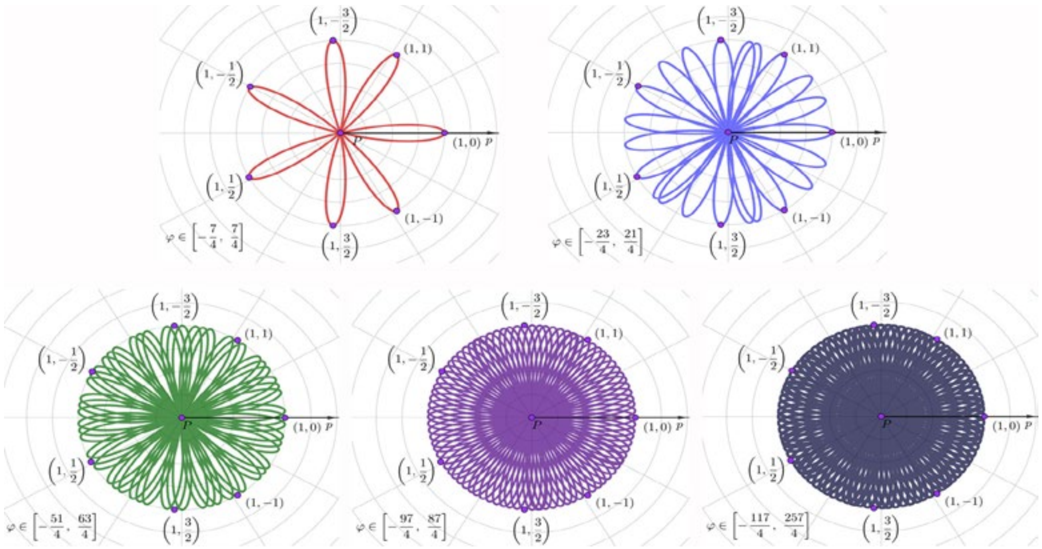

Figure 15: The Petals of The Rose Curve r = cos($2\pi\phi$) In Relation to The Different Lengths of The Intervals of The Polar Angles.

We note that the given sequence of intervals, for which the given petals are complete, are not simultaneously adjacent petals. By applying (21) we also find that 7, 22,57, 92, 187 *petals are bounded by the following intervals in the given sequence* $\left[-\frac{7}{4}, \frac{7}{4}\right]$,$\left[-\frac{23}{4}, \frac{21}{4}\right]$ $\left[-\frac{51}{4}, \frac{63}{4}\right]$, $\left[-\frac{97}{4}, \frac{87}{4}\right]$, $\left[-\frac{117}{4}, \frac{257}{4}\right]$ *, as shown in Figure 15 from top left to bottom right. Indeed, for $i = -25$ and $j = 31$, using (21), we obtain that* 57 *petals of the rose curve $r = \cos(2\pi\phi)$ are bounded by the interval* $\left[-\frac{51}{4}, \frac{63}{4}\right]$*because $j - i + 1 = 57$.*

Moreover, each of these 57 petals has the peak at $\left(1, \frac{l}{2}\right)$ *and is bounded by the interval* $\left[\frac{(2l-1)}{4}, \frac{(2l+1)}{4}\right]$*, where l is an integer such that* $-25 \leq l \leq 31$*, see* (20).

Example 9. *Another interesting example related to an irrational angular frequency is the rose curve $r = a\cos(e\phi)$ with $k = e$, where e is a mathematical constant also known as Euler's number. It is an irrational and transcendental number that is approximately equal to* 2.72. *Again, this rose curve is never complete, but we can find the interval in which some of its petals are complete. Since* $\mathrm{P} = \frac{2\pi}{e}$ *is the period of the cosine function, the petal with the peak at* $\left(1, \frac{l \cdot \pi}{e}\right)$ *is bounded by the interval* $\left[\frac{(2l-1)\pi}{2e}, \frac{(2l+1)\pi}{2e}\right]$*, where l is an arbitrary integer, see* (20)*. Moreover, by* (21) *we obtain that $j - i + 1$ petals are bounded by an interval* $\left[\frac{(2i-1)\pi}{2e}, \frac{(2j+1)\pi}{2e}\right]$ *where i, j are such integers that $i \leq j$.Then we obtain, similarly as above, that* 22, 57, 93 *petals are bounded by the following intervals in the given sequence* $\left[\frac{11\pi}{2e}, \frac{55\pi}{2e}\right]$, $\left[-\frac{51\pi}{2e}, \frac{63\pi}{2e}\right]$, $\left[-\frac{119\pi}{2e}, \frac{67\pi}{2e}\right]$ *, as shown in Figure 16 from left to right.*

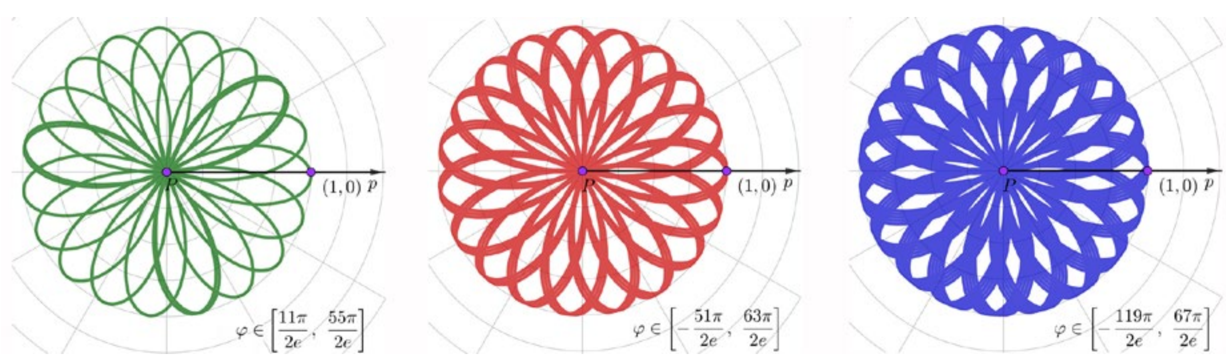

Figure 16: The petals of the rose curve $r = \cos(e\phi)$ in relation to the different lengths of the intervals of the polar angles.

We note that the number of petals does not depend on the interval, but on its length. In this way, the same number of petals is bounded by an infinite number of intervals of the same length. For different intervals of the same length, however, we obtain different petals.

4. Declarations

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References

1. Lawrence, J. D. (2013). A catalog of special plane curves. Courier Corporation.
2. O'Connor, J. J., Robertson, E. F., *Luigi Guido Grandi. MacTutor History of Mathematics archive*, University of St Andrews. Retrieved from http://wwwhistory.mcs.st-andrews.ac.uk/Biographies/Grandi.html
3. Savelov, A. A. (1979). *Plane Curves (Ravninske krivulje - in Croatian).* Skolska knjiga, Zagreb.
4. Smith, D. E. (1958). *History of Mathematics*, Volume II, Dover Publications, Inc., New York.