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Research Article

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Neuro-Psychological Interpretations of Mathematical Results Reported in Case of Discrete- Time Hopfield Neural Networks

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Abstract
In this paper, for mathematical descriptions of electrical phenomena (voltage state) appearing in nervous system discrete-time Hopfield neural network is used. The equilibrium states of a discrete-time Hopfield neural network are interpreted as equilibriums of the nervous system. An equilibrium state for which the steady state is locally exponentially stable is interpreted as robust equilibrium of the nervous system. That is because after a small perturbation of the equilibrium steady state the network recover the equilibrium. A path of equilibrium states for which the steady states are locally exponentially stable is interpreted as a path of robust equilibriums of the nervous system. This is a way to follow in healthcare for transfer gradually the nervous system from a pathologic robust equilibrium into a non-pathologic robust equilibrium. For illustration, computed way of transfer is presented.

Keywords: Discrete-Time Hopfield Type Neural Networks, Nervous System, Robust Equilibrium, Nervous System Control

MSC: 37B25; 62M45; 65P20; 92B20.

1. Introduction

A semi-discretization of the continuous-time Hopfield neural network has been made for obtain discrete-time neural networks. The result is the following discrete semi-dynamical system:

$$x_{p+1}^{i} = e^{-a_i \times h} \times x_p^i + \frac{1-e^{-a_i \times h}}{a_i} \times \left(\sum_{j=1}^{n} T_{ij} \times g_j\left(x_p^i\right) + I_i\right) \qquad i = 1,2, \dots n \quad p \in N \quad (1.1)$$

where $h > 0$ is the uniform discretization step size. It has been established that for any $h > 0$ the discrete-time neural network (1.1) faithfully preserve the characteristics of the continuous-time Hopfield neural network, i.e. the steady states and their stability properties. In Balint and..2008 pg.189 more general class of discrete- time Hopfield neural networks (which includes (1.1)) were considered. These are defined by the following discrete semi-dynamical system:

$$x_{p+1}^{i} = b_i \times x_p^i + \sum_{j=1}^{n} \widehat{T_{\iota,J}} \times g_j\left(x_p^i\right) + \widehat{I_\iota} \qquad i = 1,2, \dots n \ , \quad p \in N \qquad (1.2)$$

where , $b_i \in (0,1)$, $\widehat{I_\iota}$ denotes the external input, $\hat{T} = (\widehat{T_{\iota,J}})$ is the interconnection matrix, $g_i : R \to R$, $i = 1,2,\dots n$,represent the neuron input-output activations. The system (1.2) were analyzed in Balint and..2008. It was assumed in general that the activation functions has the following properties: $g_i(0) = 0$ for ., $i = 1, 2,\dots n.$, $| g_i(s) \le 1 |$ for any $s \in R, i = 1,2, \dots n.$,. and there exist $k_i > 0$ such that $0 < g'_i(s) < k_i$ for any $s \in R, i = 1,2, \dots n.$,.

The system (1.2) can be written in the matrix form:

$$X_{p+1} = B \times X_p + \hat{T} \times G(X_p) + \hat{I} \quad (1.3)$$

when $X_p = (x_p^1, x_p^2, \ldots, x_p^n)^T \in R^n$, $B = diag(b_1, b_2, \ldots b_n) \in M_{n\times n}$,$\hat{I} = (\widehat{I_1}, \widehat{I_2}, \ldots \widehat{I_n})^T \in R^n$ and $G: R^n \to R^n$,is ,is given by $G(X) = (\ g_1(x^1),\ g_2(x^2), \ldots\ g_n(x^n))^T$.

Using the function $F : R^n \times R^n \to R^n$ defined by $F(X, \hat{I}\ ) = B \times X + \hat{T} \times G(X) + \hat{I}$ The semi- dynamical system (1.3) can be written in the form:

$$X_{p+1} = F(X_p, \hat{I}\ ) \quad (1.4)$$

An equilibrium of the semi-dynamical system (1.4) by definition is a solution of the nonlinear equation

$$F(X, \hat{I}\ ) = X \quad (1.5)$$

This means that an equilibrium E is a couple $E = (X, \hat{I}\ ) \in R^n \times R^n$ which verifies (1.5). If $E = (X, \hat{I}\ ) \in R^n \times R^n$ is an equilibrium of (1.4) then X is called steady state and $\hat{I}$ is called external input of (1.4). The name steady state is justified by the fact that: if $(X^0, \widehat{I^0}) \in R^n \times R^n$ is an equilibrium of (1.4) then keeping in (1.4) the initial condition $X_1^0 = (x_{0,}^1 x_{0,}^2 \ldots, x_{0,}^n)^T$ and the external input $\hat{I}$ equal to $\widehat{I^0} = (\widehat{I_1^0}, \widehat{I_2^0}, \ldots \widehat{I_n^0}\ )^T$ the formula (1.4) generate a constant sequence in which each term is equal to X_1^0.

According to Balint and..2008 Theorem 5.15. pg.190. for any state $X \in R^n$ there exists an external input $\hat{I} \in R^n$ such that the couple (X, $\hat{I}$) is an equilibrium of the semi-dynamical system (1.4). The input $\hat{I}$ is given by the formula:

$$\hat{I} = (I_d - B) \times X - \hat{T} \times G(X) \quad (1.6)$$

where $I_d \in M_{n\times n}$ is the identity matrix?

On the other hand, it can happen that for the same external input $\widehat{I^0}$, there exist one, or several different voltage states X^j such that $E^{j,\widehat{I^0}} = (X^j, \widehat{I^0})$ are equilibriums for (1.4).

2. Neuro-Psychological Interpretation of the Equilibrium

From neuro-psychological point of view the mark of an equilibrium of the nervous system, described by the neural network (1.4), is the constancy of the voltage state of the neurons, providing that the external input is maintained constant. Therefore, is natural to interpret an equilibrium $E = (X, \hat{I}\ )$ of the neural network as equilibrium of the nervous system. Hence, come the idea that in order to change a pathologic equilibrium $E^{0,\widehat{I^0}} = (X^0, \widehat{I^0})$ of the nervous system, a new external electrical input $\widehat{I^1}$ has to be applied. If the steady voltage state of the new non- pathologic equilibrium is $X^1 = (x_1^1, x_1^2 \ldots x_1^n)^T$ then it is natural to think that the new external electrical input $\widehat{I^1}$,which has to be applied, has to be taken according to the formula (1.6), hoping that, after the external electrical input change $\widehat{I^0} \to \widehat{I^1}$, the pathologic steady voltage state $X^0 = (x_0^1, x_0^2, \ldots, x_0^n)^T$, of the nervous system, evolve to the non-pathologic steady voltage state $X^1 = (x_1^1, x_1^2 \ldots x_1^n)^T$. Mathematically this neuro-psychological though is correct if the solution of the initial value problem

$$X_{p+1} = F(X_p, \widehat{I^1}\ ) \quad , \quad X_1 = X^0 = (x_0^1, x_0^2, \ldots, x_0^n)^T \quad (2.1)$$

tends to the steady voltage state $X^1 = (x_1^1, x_1^2 \ldots x_1^n)^T$.

This kind of reasoning make sense if $\widehat{I^0} \neq \widehat{I^1}$. That is because, if $\widehat{I^0} = \widehat{I^1}$ then there is no change in input and the voltage state of the neural network will rest in the state X^0i.e. the voltage state evolution of the neural network is described by (2.1) is constant equal to X^0.

Moreover, even if $\widehat{I^0} \neq \widehat{I^1}$and the reasoning make sense, it can happen that for the new electrical input $\widehat{I^1}$ beside the non-pathologic voltage state X^1, there exist a second voltage state X^2,and applying the electrical input $\widehat{I^1}$ beside the non-pathologic equilibrium $E^1 = (\ X^1\ , \widehat{I^1}\ )$ a second equilibrium $E^2 = (X^2, \widehat{I^1})$ appear. It can happen that the equilibrium $(X^2, \widehat{I^1})$ is pathologic too. Therefore, the problem is to find supplementary condition assuring that the solution of the initial value problem (2.1) tends to X^1 a s it was planned.

In Balint and. 2008 example (5.5) pg.197 provide computational simulation of the above-described phenomena.

Consider the discrete semi-dynamical system, obtained from the continuous-time system analyzed in Balint and..2008 example 5.2 pg.184, by the semi-discretization technique.

$$x_{p+1}^{1} = e^{-h} \times x_{p}^{1} + (1 - e^{-h}) \times (\frac{17 \times ln4}{15} \times tanhx_{p}^{2} + \widehat{I_1^1}) \text{ , } x_{p+1}^{2} = e^{-h} \times x_{p}^{2} + (1 - e^{-h}) \times (\frac{17 \times ln4}{15} \times tanhx_{p}^{1} + \widehat{I_2^1}) \quad (2.2)$$

For $h = 0.2$, and $\widehat{I^0} = (0,0)^T$, for (2.2) the following steady states were found: $X^0 = (0,0)^T$, $X^1 = (ln4, ln4)^T$, $X^2 = (-ln4, -ln4)^T$.

If the voltage of the neural network is in the steady state $X^0 = (0,0)^T$ and the value of the external input is maintained $\widehat{I^0} = (0,0)^T$ then the voltage rest constant. This phenomenon is illustrated on figures 2.1 and 2.2

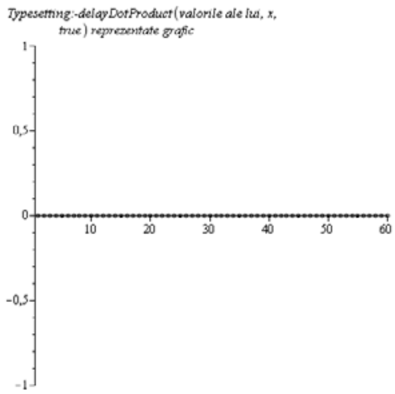


Figure: 2.1.x_1 versus p in $E^0 = (X^0, \widehat{I^0})$

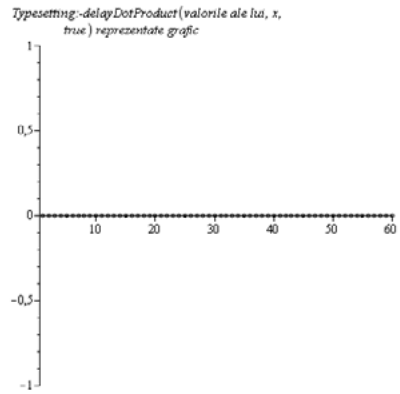


Figure: 2.2.x_2 versus p in $E^0 = (X^0, \widehat{I^0})$

These figures show that maintaining the external input value $\widehat{I^0} = (0,0)^T$, the voltage of the neural network is constant equal to $X^0 = (0,0)^T$.

According to the neuro-psychological interpretation, this type of the neural network voltage behavior indicates that $E^0 = (X^0, \widehat{I^0})$ is an equilibrium of the corresponding nervous system.

Assume that the equilibrium $E^0 = (X^0, \widehat{I^0})$ is non -pathologic and the equilibrium $E^3 = (X^3, \widehat{I^3})$,with $X^3 = (0.1,0.1)^T$ and $\widehat{I^3} = (-0.1565911736, -0.1565911736)^T$,is pathologic and a neurological or psychological intervention is needed. The change of the external electrical input represents a possible intervention. Assume that the medical decision is to transform the pathologic equilibrium $E^3 = (X^3, \widehat{I^3})$ into the equilibrium $E^0 = (X^0, \widehat{I^0})$ by changing the external electrical input $\widehat{I^3} = (-0.1565911736, -0.1565911736)^T \to I^0 = (0,0)^T$ at the moment of time $p_1 = 0$. The effect of the external input change $\widehat{I^3} \to \widehat{I^0}$ is represented on figures 2.3 and 2.4.

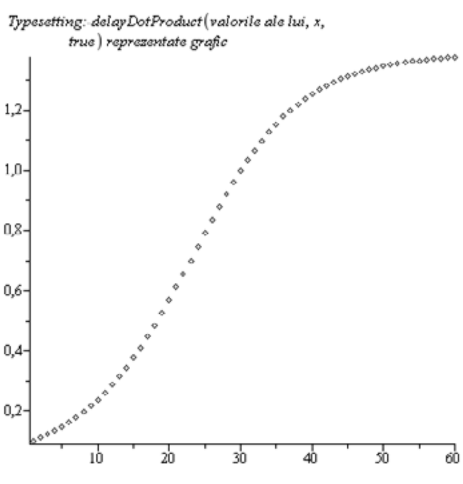


Figure: 2.3.x_1 versus p

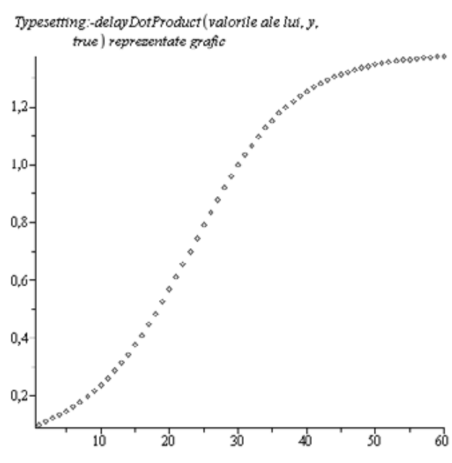


Figure: 2.4.x_2 versus p

These figures show that after the input change the pathologic voltage state $X^3 = (0.1,0.1)^T$ do not evolve to the non-pathologic voltage state voltage state $X^0 = (0,0)^T$ as it was expected. The mathematical explanation is: the steady state $X^1 = (ln4, ln4)^T$ is locally exponentially stable and the steady state $X^3 = (0.1,0.1)^T$ belongs to the region of attraction of the steady state $X^1 = (ln4, ln4)^T$.

In the same time, the steady voltage state $X^0 = (0,0)^T$ is unstable and repulsive.

The unstable character of steady voltage state means that for any small perturbation of the initial condition, the solution of the perturbed initial value problem:

$$x_{p+1}^1 = e^{-h} \times x_p^1 + (1 - e^{-h}) \times (\frac{17 \times ln4}{15} \times tanhx_p^2) \qquad x_1^1 = \varepsilon$$
$$x_{p+1}^2 = e^{-h} \times x_p^2 + (1 - e^{-h}) \times (\frac{17 \times ln4}{15} \times tanhx_p^1) \qquad x_1^2 = \delta \qquad (2.3)$$

do not recover the steady state $X^0 = (0,0)^T$.

The next figures illustrate the instability and the repulsive character of the equilibrium .

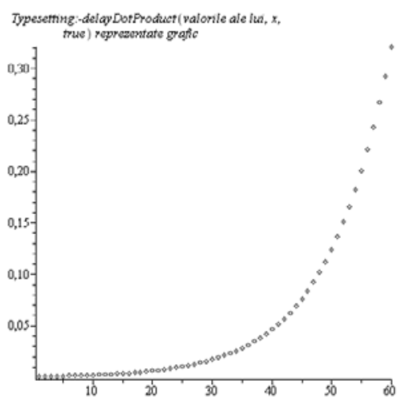


Figure: 2.5.x_1 versus p

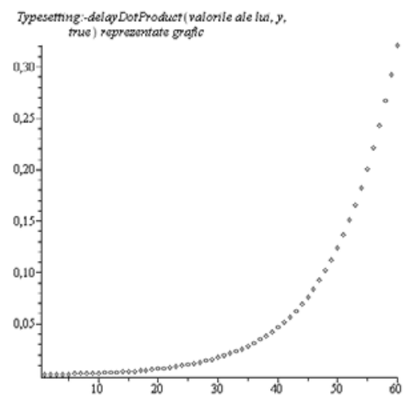


Figure: 2.6.x_2 versus p

Figures 2.5 ,2.6 illustrate that instead of the recovery of $X^0 = (0,0)^T$the components of the perturbed steady state X^0 move away from the steady state $X^0 = (0,0)^T$.

Figures 2.5.,2.6 illustrate also the repulsive character of the steady voltage state $X^0 = (0,0)^T$. That is because the solution of the initial value problem (2.3) represent also the evolution of the steady state $X^{\varepsilon,\delta} = (\varepsilon, \delta)^T$ of the equilibrium $E^{\varepsilon,\delta} = (X^{\varepsilon,\delta}, \widehat{I^{\varepsilon,\delta}})$ in case of the external electrical impulse change $\widehat{I^{\varepsilon,\delta}} \rightarrow (0,0)^T$.The external electrical input $\widehat{I^{\varepsilon,\delta}}$ appearing here is obtained from the steady state $X^{\varepsilon,\delta}$ using formula (1.6).

According to the neuro-psychological interpretation of equilibrium, is important to keep in mind that in a nervous system there are three types of equilibriums:

-Equilibriums of first type for which after a small perturbation of the steady state, the nervous system return to the equilibrium. This is the situation if the steady state of the corresponding neural network is locally exponentially stable. (as is the equilibrium E^1 =($(ln4,ln4)^T$, $\widehat{I^0}$=$(0,0)^T$)).Due to this property the nervous system return to the equilibrium automatically ,without any external input, we will say that this equilibrium of the nervous system is robust.

-Equilibriums of second type for which after a small perturbation of the steady state the nervous system do not return to the equilibrium. This is the situation if the steady state of the corresponding neural network is unstable. (as is the equilibrium E^0=(X^0,$\widehat{I^0}$) with X^0=$(0,0)^T$ and $\widehat{I^0}$=$(0,0)^T$) . Due to the property that the nervous system do not return to the equilibrium automatically, without applying an external input, we will say that this type of equilibrium of the nervous system is fragile.

--Equilibriums of third type having the property that there is no equilibrium, which can be transferred in such type of equilibrium. Due to this property, we will say that this type of equilibrium of the nervous system is repulsive.((as is the equilibrium E^0=(X^0,$\widehat{I^0}$) with X^0=$(0,0)^T$ and $\widehat{I^0}$=$(0,0)^T$)).

3. Equilibriums Transfer

A correct neuro psychological interpretation and understanding of the possible equilibriums of the neural network permit to neurologist and psychologist to choose appropriate tool in a specific case. On this basis people, working in neural and mental healthcare, can choose appropriate tool for transfer the pathologic equilibrium of a patient into a non-pathologic equilibrium.

The choice of the appropriate tool assume : starting from a robust pathologic equilibrium $E^0 = (X^0, \widehat{I^0})$, choose a new non-pathologic steady state X^1, compute for X^1 , the corresponding new external electrical input $\widehat{I^1}$ using formula (1.6), and build up a new robust non-pathologic equilibrium $E^1 = (X^1, \widehat{I^1})$.

After that, several computations has to be made in order to be able to transfer $E^0 = (X^0, \widehat{I^0}) \rightarrow E^1 = (X^1, \widehat{I^1})$.

Step 1: Verify that the equilibriums E^0, E^1 are robust and the region of attraction of the steady state X^1 contains the steady state X^0. A way to verify the robustness of the equilibriums E^0, E^1 is to solve and represent the solutions of the initial value problems:

$$X_{p+1} = F(X_p, \widehat{I^0}) \qquad X_1 = X_1^0 \tag{3.1}$$

$$X_{p+1} = F(X, \widehat{I^1}) \qquad X_1 = X_1^1 \tag{3.2}$$

where X_1^0 X_1^1, are small perturbations of X^0 and X^1 respectively.

Step 2: Solve and represent the solutions of the initial value problems

$$X_{p+1} = F(X, \widehat{I^0}) \qquad X_1 = X^1 \tag{3.3}$$

$$X_{p+1} = F(X, \widehat{I^1}) \qquad X_1 = X^0 \tag{3.4}$$

In order to see how this work in practice, consider the neural network (2.2) and the equilibrium $E^0 = (X^0, \widehat{I^0}) = ((ln4, ln4)^T, (0,0)^T) = (1.386294361, 1.386294361)^T, (0,0)^T)$.

Fix the new steady state $X^1 = (0.1 + ln4, 0.1 + ln4)^T = (1.486294362, 1.486294362)^T$ and using (1.6) compute the corresponding new external electrical input $\widehat{I^1}$ finding $\widehat{I^1} = (0.068125374, 0.068125374)^T$.

So, the new equilibrium is

$$E^1 = (X^1, \widehat{I^1}) = (1.486294362, 1.486294362)^T, (0.068125374, 0.068125374)^T).$$

For test the robustness of $E^0 = (X^0, \widehat{I^0})$,solve and represent the initial value problem (3.1). Taking for example $X_1^0 = (1.486294362, 1.486294362)^T$ the solution of the initial value problem (3.1) is presented on the next figures:

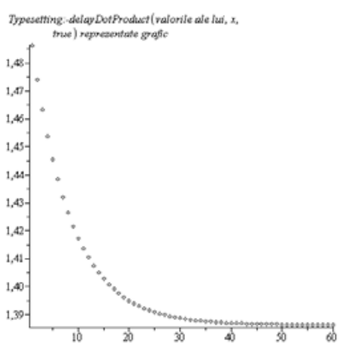


Figure: 3.1.x_1 versus p

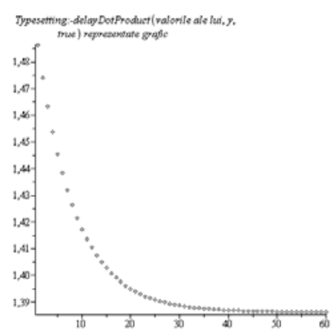


Figure: 3.2.x_2 versus p

Figures 3.1. and 3.2. suggest that the perturbed steady state X^0 in case of the equilibrium $E^0 = (X^0, \widehat{I^0})$ retrieve the steady state. In other words, these figures suggest that the equilibrium E^0 is robust. Moreover, these figures illustrate also the transfer of the equilibrium E^1 into the equilibrium E^0, because they represent solution of the initial value problem (3.3)
The robustness of the equilibrium E^1 and the transfer of E^0 into E^1 can be illustrate solving the initial value problem (3.2) for $\widehat{I^1} = (0.068125374, 0.068125374)^T$ and $X_1^1 = (1.386294361, 1.386294361)^T$. The solution of (3.2) in this case is presented in the next figures.

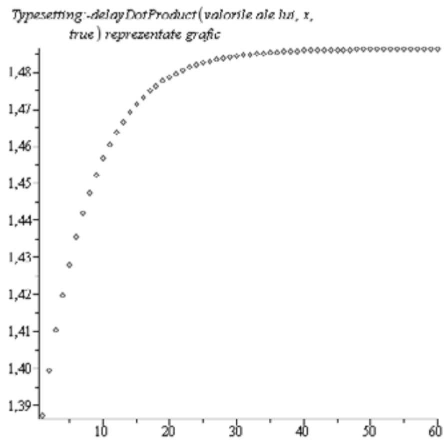


Figure: 3.3.x_1 versus p

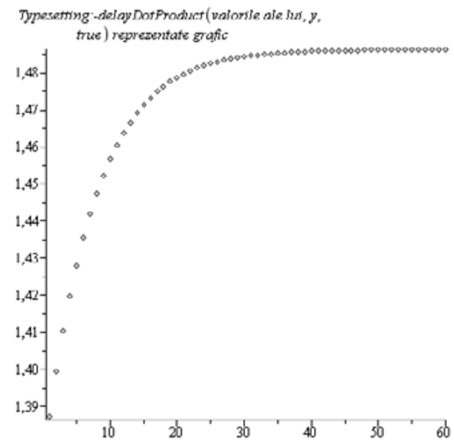


Figure: 3.4.x_2 versus p

In the following, we present results from Balint and..2008, which can offer an overview about the complexity of the possible equilibriums, their location, the steady states character and possible transfer. Illustrative computational examples are given.

In Balint and..2008 theorem 5.17.pg.191 states: If Δ is a rectangle in R^n, (i.e. for i = 1,2,..n there exist $\alpha_i, \beta_i \in R \quad \alpha_i < \beta_i$, such that $\Delta = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2) \times \ldots (\alpha_n, \beta_n))$ and det $\left((I_d - B) - \widehat{T} \times D_X(X)\right)\right) \neq 0$ for any $X \in \Delta$, then

- The restriction of the function $\widehat{I_\Delta}$ (the restriction to Δ of the external input function) is injective.
- For any input $\hat{I} \in \widehat{I_\Delta}(\Delta)$ the system (1.5) has a unique steady state in Δ.
- If 0 < g'_i(s) ,for any $s \in R, i = 1,2, \ldots n$, and

$$\widehat{T_{\iota\iota}} - \frac{1-b_i}{g_i'(x_i)} + \sum_{j \neq i} \left|\widehat{T_{J\iota}}\right| < 0 \qquad i = 1,2, \ldots n \qquad X \in \Delta \qquad , \tag{3.4}$$

then for any $\hat{I} \in \widehat{I_\Delta}(\Delta)$ the neural network (1.5) has a unique steady state in Δ. Moreover if $\widehat{I_\iota} >$0 for any i = 1,2,...n then the coordinates of the steady voltage state are positive.

This theorem reveal that in a prior given rectangle Δ (included in R^n) if det $\left((I_d - B) - \widehat{T} \times D_X(X))\right) \neq 0$ for any $X \in \Delta$, then for any $X^0 \in \Delta$ the input $\hat{I}(X^0) = (I_d - B) \times X^0 - \hat{T} \times G(X^0) = \widehat{I^0}$ is unique. Therefore, the equilibrium $E^0 = (X^0, \widehat{I^0})$ of the nervous system is unique. This situation is completely different from that described in case of the neural network (2.2) where in case of the rectangle $\Delta = (-1.5, 1.5) \times (-1.5, 1.5)$ for the input $\widehat{I^0} = (0,0)^T$ three different equilibriums, $E_1^0 = ((0,0)^T, \widehat{I^0})$, $E_2^0 = ((ln4, ln4)^T$ $\widehat{I^0})$ and $E_3^0 = ((-ln4, -ln4)^T, \widehat{I^0})$ exists each of them having the steady state in Δ.

In Balint and 2008 theorem5.18.pg.191. states; Under the general hypothesis concerning activation functions, for any external input $\hat{I} \in R^n$ the following statements hold:

- There exits at least one steady voltage state of the neural network (1.4) (corresponding to $\hat{I}$) in the rectangle $\Delta = [-M_1, M_1] \times [-M_2, M_2] \times \ldots \times [-M_n, M_n]$

where

$$M_i = \frac{1}{1-b_i} \times \left(\left|\widehat{I_\iota}\right| + \sum_{j=1}^{n} \left|\widehat{T_{\iota J}}\right|\right) \qquad \text{for any} \quad i = 1,2, \ldots n \tag{3.5}$$

- Every steady voltage state of the neural network (1.4) corresponding to $\hat{I}$, belongs to the rectangle Δ defined above.
- If in addition if

$$\det\left((I_d - B) - \hat{T} \times D_X(X)\right) \neq 0 \text{ for any } X \in \Delta \quad (3.6)$$

then the neural network (1.4) has a unique steady state, corresponding to $\hat{I}$, and it belongs to Δ.

This theorem clarifies several things:

- First, the theorem assures that applying an arbitrary prior given external electrical input $\widehat{I^0} \in R^n$ to the nervous system there exist at least one steady voltage state X^0 such that $E^0 = (X^0, \widehat{I^0})$ is an equilibrium of the nervous system. The steady voltage state X^0 of the equilibrium $E^0 = (X^0, \widehat{I^0})$ is in the rectangle Δ specified above. This is in fact a localization of the steady voltage state. For find effective, the steady voltage state X^0, the nonlinear algebraic equation $F(X, \widehat{I^0}) = X$ has to be solved in Δ.
- Second, the theorem assures that applying an arbitrary prior given external electrical input $\widehat{I^0} \in R^n$ to the nervous system, every steady voltage state X^0 which appear due to that is in the rectangle Δ.
- Third if $\det\left((I_d - B) - \hat{T} \times D_X(X)\right) \neq 0$ for any $X \in \Delta$, (specified above) then the obtained steady voltage state X^0 is unique. This means that the obtained equilibrium $E^0 = (X^0, \widehat{I^0})$ is unique. The supplementary information is that X^0 is unique and the equilibrium $E^0 = (X^0, \widehat{I^0})$ is unique.

In Balint and…2008 theorem 5.20. pg.192 states: Under the general hypothesis concerning activation functions g_i if

$$g_i(s) = 1 \; if \;\; s \geq 1 \qquad \text{and} \qquad g_i(s) = -1 \; if \;\; s \leq -1 \quad (3.7)$$

then for any input $\hat{I} \in R^n$ satisfying

$$\left|\hat{I^i}\right| < \widehat{T_{ii}} + b_i - 1 - \sum_{i \neq j} \left|\widehat{T_{ij}}\right| \qquad \text{for} \quad \text{any} \quad i = 1,2, \dots n \quad (3.8)$$

The following statements hold:

- In every rectangle Δ_ε, $\varepsilon \in \{\pm 1\}$ there exists a unique steady voltage state of the neural network (1.4) corresponding to $\hat{I}$; here $\Delta_\varepsilon = J(\varepsilon_1) \times J(\varepsilon_2) \dots J(\varepsilon_n)$, $J(-1) = (-\infty, -1)$, , $J(1) = 1$, ∞). .
  every $\overline{\Delta_\varepsilon}$, $\varepsilon \in \{\pm 1\}$, is invariant to the map $X \to F(X, \hat{I})$.
- The mathematical condition (3.8) concerns the magnitude of the external input (left hand side) and the coefficients of the neural network (right hand side). If an input $\hat{I}^0$ which verifies (3.8) is applied to the nervous system then, due to that in the nervous system, 2^n $equilibriums \; E^{\varepsilon, \hat{I}^0} = (X^{\varepsilon, \hat{I}^0}, \hat{I}^0) appear$. Each steady state $X^{\varepsilon, \hat{I}^0}$ is unique and located in a rectangle Δ_ε. This is an extremely complex configuration of steady voltage states of the nervous system which can appear after applying an external electrical input $\hat{I}^0$.

A modified variant of the above theorem is Theorem (5.21) pg.193 [1].

In Balint and…2008 theorem5.21.pg.193. states; Under the general hypothesis concerning activation functions if there exists $\alpha \in (0,1)$ such that the activation functions verify:

$$g_i(s) \geq \alpha \; if \;\; s \geq 1 \quad \text{and} \quad g_i(s) \leq -\alpha \; if \;\; s \leq -1 \quad \text{for any} \quad i = 1,2, \dots n \quad (3.9)$$

then for any input $\hat{I} \in R^n$ satisfying

$$\left|\widehat{I_i}\right| < \widehat{T_{ii}} \times \alpha + b_i - 1 - \sum_{i \neq j} \left|\widehat{T_{ij}}\right| \quad \text{for any} \quad i = 1,2, \dots n \quad (3.10)$$

the following statements hold:

- In every rectangle Δ_ε, $\in \{\pm 1\}$, there exists a unique steady voltage state of the neural network (1.4) corresponding to $\hat{I}$.
  every Δ_ε, $\in \{\pm 1\}$, is invariant to the map $X \to F(X, \hat{I})$.
- This theorem reveal that if the neuron input-output activations verify (3.9) and one input $\widehat{I^0}$,which verifies (3.10), is applied to the nervous system then due to that in the nervous system 2^n $equilibriums \; E^{\varepsilon, \widehat{I^0}} = (X^{\varepsilon, \widehat{I^0}}, \widehat{I^0}) appear$.Each steady state $X^{\varepsilon, \widehat{I^0}}$ is unique and located in a rectangle Δ_ε. This configuration of steady states, is similar which appear in theorem 5.20 and is an extremely complex configuration of steady voltage states of the nervous system which can appear after applying an external electrical input $\widehat{I^0}$.

Finally, in Balint and…2008 theorem5.25.pg.195 conditions of local exponential stability of the above steady voltage states are presented.

In Balint and…2008 theorem5.25.pg.195. states; suppose that the conditions of Balint and…2008 theorem 5.21 pg.193 are fulfilled. Let be an input $\widehat{I^0} \in R^n$ satisfying (3.10) and $\varepsilon \in \{\pm 1\}$,. If $|g_i'(s)| < \frac{1-b_i}{\sum_{j=1}^{n}|\widehat{T_{ji}}|}$ for any and $|s| \geq 1$ and $i = 1,2,\dots n$ then the steady voltage state $X^{\varepsilon,\widehat{I^0}}$ of the neural network (1.4) corresponding to $\widehat{I^0}$, which lies in the rectangle Δ_ε, is unique it is locally exponentially stable and its region of attraction includes $\overline{\Delta_\varepsilon}$.

This theorem present conditions, concerning the neuron input-output activations, assuring the robustness of the 2^n $equilibriums$ $E^{\varepsilon,\widehat{I^0}} = \left(X^{\varepsilon,\widehat{I^0}}, \widehat{I^0}\right)$ $which\ appear\ after\ applying\ the\ external\ electrical\ input$ $\widehat{I^0}$ to the nervous system.

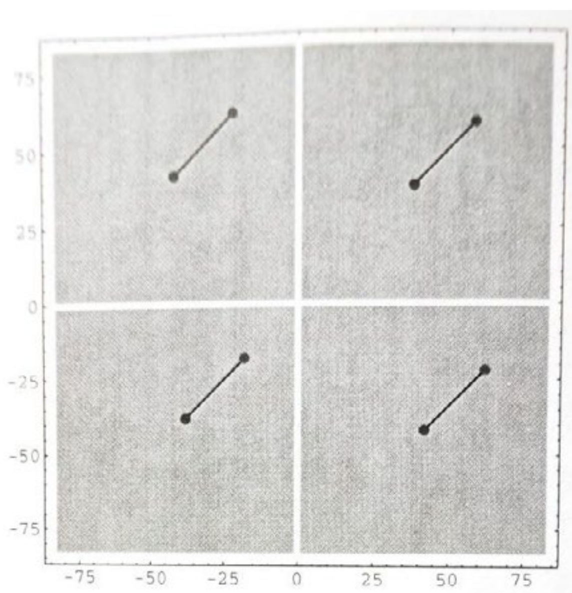
The importance of this theorem consists in the fact that permit the construction of path of robust equilibriums which can be used in healthcare as secured way to transfer gradually a pathologic equilibrium into a non-pathologic equilibrium.

A numerical illustration of the phenomena described in Theorem 5.25. pg.195. Balint and (2008) is given in Example (5.6) pg.197 Balint and (2008). In this example the following Hopfield neural network is considered:

$$x_{p+1}^1 = 0.5x_p^1 + 20f\left(x_p^1\right) - f\left(x_p^2\right) + \widehat{I_1} \quad , \quad x_{p+1}^2 = 0.5x_p^2 - f\left(x_p^1\right) + 20f\left(x_p^2\right) + \widehat{I_2} \qquad (3.11)$$

With the non-monotone activation function $f(x) = \tanh(x)\tanh(10x^2 - 1)$. It has been shown that in some cases, the absolute capacity of an associative neural network can be improved by using non-monotone activation functions instead of the usual sigmoid ones. The conditions of theorems Balint and…2008 theorem5.21.pg.193 and theorem5.25.pg.195 are verified ($\alpha = f(1) \in (0,1)$). Therefore , for any input $\hat{I} = (\widehat{I_1}, \widehat{I_2})^T$ such that $|\widehat{I_i}| < 18.4982$ there exists unique locally exponentially stable steady state $X^{\varepsilon,\hat{I}} = (x^{1;\varepsilon,\hat{I}}, x^{2;\varepsilon,\hat{I}})^T$ in each rectangle Δ_ε.

In the next figure the rectangles represent the sets $S_\varepsilon = \{\frac{x^{i;\varepsilon,\widehat{I_i}}}{|\widehat{I_i}|} < 18.4982, i = 1,2\}$



Sets S_ε

The four steady states corresponding to the input $\hat{I} = (0,0)^T$ are: $(38,38)^T$, $(-42,42)^T$, $(42,-42)^T$, $(-38,-38)^T$
The four steady states corresponding to the input $\hat{I} = (10,10)^T$ are: $(58,58)^T$, $(-42,42)^T$, $(-22,62)^T$, $(62,22)^T$, $(-18,18)^T$

The external electrical input change $(0,0)^T \rightarrow (10,10)^T$ transfer the configuration of steady states { $(38,38)^T$, $(-42,42)^T$, $(42,-42)^T$, $(-38,-38)^T$} into the configuration of steady states { $(58,58)^T$, $(-42,42)^T$, $(-22,62)^T$, $(62,22)^T$, $(-18,18)^T$ }

The transfer of the steady state voltage $(38,38)^T$ into the steady state voltage $(58,58)^T$ due to the external electrical input change $(0,0)^T \rightarrow (10,10)^T$ is presented in the next figures.

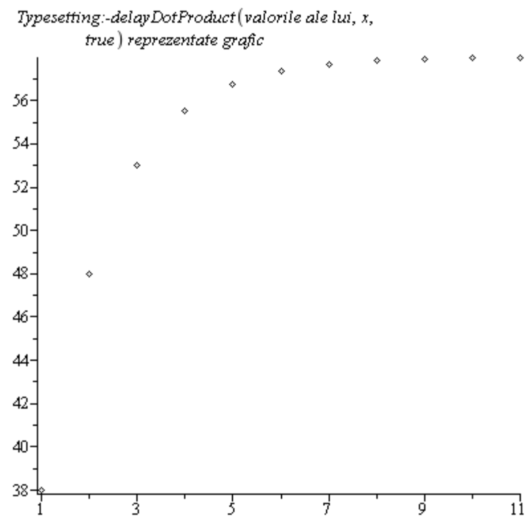


Figure: 3.5.x_1 versus p

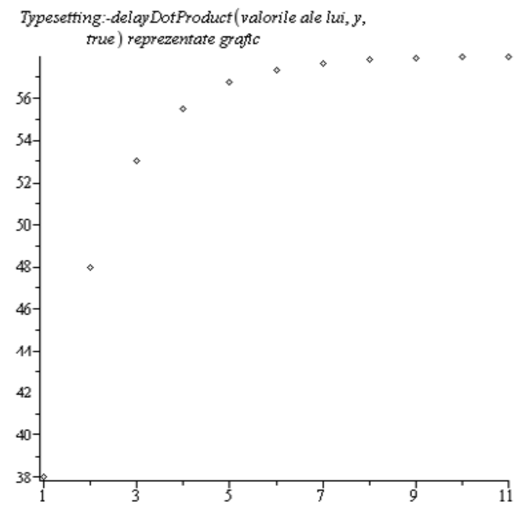


Figure: 3.6.x_2 versus p

This example intends to illustrate the nervous system equilibriums configuration complexity, the existence of robust equilibrium paths and the equilibrium steady voltage states transfer along the equilibrium pats. The strategy in computational neurology has to be the buildup path of locally exponentially stable steady states along which by small successive changes, the neural network voltage can be conducted, through the regions of attraction of intermediary locally exponentially stable steady voltage states from a pathologic steady voltage state to a final non-pathologic steady voltage state. This kind of interpretation of the mathematical results, obtained in the discrete-time Hopfield neural network model, can be useful in healthcare for establish safe neuro-psychological treatment procedures.

4. Conclusion

The mathematical theory of discrete-time Hopfield neural networks, presented in this paper exhibit the same scale of scenarios as the continuous-time Hopfield neural networks having similar neuro psychological interpretation. Robust, fragile, and repulsive equilibriums appear. In equilibrium, the voltage state of the neural network is constant and does not change if the external electrical input value is maintained constant. Equilibrium transfer analysis show that the strategy in computational neurology has to be the buildup of path of robust equilibrium states along which by small successive changes, the neural network voltage can be conducted from a pathologic equilibrium, through the regions of attraction of intermediary robust equilibriums to the final non-pathologic equilibrium. In healthcare a treatment procedure usually follows a path of robust equilibriums, which "connect" the present pathologic equilibrium with a future non-pathologic equilibrium of the patient.

References

1. Balint, Ș., Brăescu, L., & Kaslik, E. (2008). Regions of attraction and applications to control theory (Vol. 1). *Cambridge Scientific Publishers.*