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Research Article

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How to Solve the Schrödinger Equation and the Klein-Gordon Equation

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Abstract
I will show how to solve the Schrödinger equation and the Klein-Gordon equation by using hyper-exponential function of second-order.

1. The Schrödinger Equation.

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(x,y,z)\psi\ .\ ---①$$

I set the following:

$$r := \{lx + my + nz \,|\, (x,y,z) \in R^3,\ l^2 + m^2 + n^2 = 1\},$$

$$\Phi(r) = exph_j^2\{r, f(r)\}\ \ (j = 0,1)\ ,$$

$$\psi(x,y,z,t) = e^{i\hbar t}\Phi(r)\ .$$

$$\frac{\partial^2\phi}{\partial x^2} = l^2\frac{d^2\phi}{dr^2}$$

$$\frac{\partial^2\phi}{\partial y^2} = m^2\frac{d^2\phi}{dr^2}$$

$$\frac{\partial^2\phi}{\partial z^2} = n^2\frac{d^2\phi}{dr^2}$$

Then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi(r) = (\ l^2 + m^2 + n^2\ )\frac{d^2\Phi}{dr^2} = \frac{d^2\Phi}{dr^2} = f(r)\Phi(r)\ .$$

Therefore,

$$-\frac{\hbar^2}{2m}\nabla^2\psi = -\frac{\hbar^2}{2m}\nabla^2\left\{e^{i\hbar t}\Phi(r)\right\} = -\frac{\hbar^2}{2m}e^{i\hbar t}f(r)\Phi(r)\ .\ ---②$$

Therefore,

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar\frac{\partial}{\partial t}\left\{e^{i\hbar t}\Phi(r)\right\} = -\hbar^2 e^{i\hbar t}\Phi(r)\ .\ ---③$$

From ①,② *and* ③,

$$-\hbar^2 e^{i\hbar t}\Phi(r) = -\frac{\hbar^2}{2m}e^{i\hbar t}f(r)\Phi(r) + V(x,y,z)e^{i\hbar t}\Phi(r)\ ,$$

$$-\hbar^2\Phi(r) = -\frac{\hbar^2}{2m}f(r)\Phi(r) + V(x,y,z)\Phi(r)\ ,$$

$$\left\{V(x,y,z) - \frac{\hbar^2}{2m}f(r) + \hbar^2\right\}\Phi(r) = 0\ .$$

Therefore,

$$If\ \Phi(r) \neq 0, V(x,y,z) - \frac{\hbar^2}{2m} f(r) + \hbar^2 = 0\,.$$

Therefore,

$$f(r) = 2m\left\{\frac{1}{\hbar^2}V(x,y,z) + 1\right\}.$$

$\therefore$

$$If\ \ V(x,y,z) = V(r),$$

$$\psi(x,y,z,t) = e^{i\hbar t}\left[\,C_1 exph_0^2\{r, f(r)\} + C_2 exph_1^2\{r, f(r)\}\,\right],$$

$$f(r) = 2m\left\{\frac{1}{\hbar^2}V(r) + 1\right\}\ .$$

2. The Klein-Gordon Equation.

$$\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2} - \frac{m^2c^2}{\hbar^2}\right)\psi = 0\,. - - - ④$$

I set the following:

c *is constant* , v *is constant* , $0 < v < c$,

$r := \{lx + my + nz \pm vt \,|\, (x,y,z,t) \in R^4,\ l^2 + m^2 + n^2 = 1\}$,

$\psi = \psi(r)$

$\psi(r) = exph_j^2\{r, f(r)\}\ \ (j = 0, 1)\,.$

$$\frac{\partial^2\psi}{\partial x^2} = l^2\frac{d^2\psi}{dr^2}$$

$$\frac{\partial^2\psi}{\partial y^2} = m^2\frac{d^2\psi}{dr^2}$$

$$\frac{\partial^2\psi}{\partial z^2} = n^2\frac{d^2\psi}{dr^2}$$

$$\frac{\partial^2\psi}{c^2\partial t^2} = \frac{v^2}{c^2}\frac{d^2\psi}{dr^2}$$

Therefore,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{c^2\partial t^2}\right)\psi(r) = \left(\ l^2 + m^2 + n^2 - \frac{v^2}{c^2}\right)\frac{d^2\psi}{dr^2} = \left(1 - \frac{v^2}{c^2}\right)f(r)\psi(r)\,.$$

From ④,

$$\left\{\left(1 - \frac{v^2}{c^2}\right)f(r) - \frac{m^2c^2}{\hbar^2}\right\}\psi(r) = 0\,.$$

Therefore,

$$If\ \psi(r) \neq 0, \left(1 - \frac{v^2}{c^2}\right)f(r) - \frac{m^2c^2}{\hbar^2} = 0\,.$$

Therefore,

$$f(r) = \frac{1}{1 - \frac{v^2}{c^2}}\frac{m^2c^2}{\hbar^2}\,.$$

$\therefore$

$$If\ \ \psi(x,y,z,t) = \psi(r),$$

$$\psi(x,y,z,t) =\ C_1 exph_0^2\{r, f(r)\} + C_2 exph_1^2\{r, f(r)\},$$

$$f(r) = \frac{1}{1 - \frac{v^2}{c^2}}\frac{m^2c^2}{\hbar^2}\,.$$

Therefore,

$$\psi(x,y,z,t) = C_1 \cosh\left(\sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}\frac{mc}{\hbar}\, r\right) + C_2 \sinh\left(\sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}\frac{mc}{\hbar}\, r\right).$$

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