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Short Article

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Hausdorff Spaces in Bermejo Algebras: The Birth of Treonic Manifold Construction

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Abstract
We conducted a topological analysis of the novel, non-associative, and unital algebraic structure known as Bermejo Algebras, developed by Alejandro Bermejo, which includes the Algebra B and Treon Algebra. This algebras can define Lie and Malcev algebras when their product operations are derived from the Bermejo Algebras product. Central to this study are treons, complex entities that arise as isomorphisms of Algebra B when the field is real. We define the vector spaces associated with Bermejo Algebras to establish equivalence classes and a quotient topology, resulting in Hausdorff spaces that do not depend on traditional norms, inner products, or metrics. Our findings lay the groundwork for constructing new types of differential manifolds, opening new avenues for advanced mathematical research.

Keywords: Bermejo Algebra, Treons, Topology, Hausdorff Space, Differential Manifolds, Non- Associative Algebras

1. Introduction

Bermejo Algebras are a novel non-associative and unital algebraic structure, founded and developed by Alejandro Bermejo [1,2]. This algebraic construction can define Lie and Malcev algebras when their product operations are derived from the product operation of Bermejo Algebras. These algebras yield complex entities, such as the algebra C^2 of vectors (a_1, a_2), which can be considered a trivial case of Bermejo Algebra: $(a_1, a_2, 0)$. These entities are distinct from quaternions and other higher hypercomplex structures [2]. Bermejo termed these complex entities ”treons”, which manifest as isomorphisms of his algebra when the field is real (R) [2]. We undertake a topological analysis of Bermejo Algebras, redefining the fundamental space of these algebras to develop a structure based solely on the products of Bermejo Algebras. By avoiding the need to define norms, inner products, or metrics, we successfully establish equivalence classes and a quotient topology that forms a Hausdorff space. Given that Hausdorff spaces are crucial in the definition of manifolds [3,4], our work lays the essential groundwork for constructing treonic differential manifolds.

This work marks the beginning of constructing a framework to support any complex analysis of the spaces discussed here. For instance, the Cauchy-Riemann equations for treons [5] will be better defined with the vector bases established here.

2. Bermejo Algebras

Bermejo algebras (Algebra B [1] and Treon Algebra [2]) can be expressed both as points in R^3, for the case (a_1, a_2, a_3), or as points in a 3-dimensional ”complex space C^3”, for the case $a_1 + a_2i + a_3j$.

The algebra B isomorphic to the treon algebra necessarily implies the definition of algebra B over the real field R; therefore, the algebra B isomorphic to the treonic algebra uses R_3 as a vector space and associates it with a product. Consequently, algebra B isomorphic to treon algebra is an algebra over R_3 and, logically, treon algebra is an algebra over R_3 [2].

Now, if we assume that in Bermejo algebras there are complex quantities [1, 2, 5], we can define a vector space with a subtly different base structure, i.e., the canonical basis of R^3 can be defined from bases in a space we will call B-Spaces:

Let $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ be the canonical base in $\mathbb{R}^3$, we define:

$$\hat{\imath} \equiv (1,0,0), \quad \hat{\jmath} \equiv (0,1,0), \quad \hat{k} \equiv (0,0,1)$$

where $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is the basis for the space associated with algebra B [1, 2]: Space B_1. And:

$$(1, 0, 0) \equiv \text{id}, \qquad (0, 1, 0) \equiv i, \qquad (0, 0, 1) \equiv j$$

where $\{\text{id}, i, j\}$ is the basis in the space associated with treon algebra [2]: Space B_2. And, by transitivity of the equivalence ≡:

$$\hat{i} \equiv \text{id}, \quad \hat{j} \equiv i, \quad \hat{k} \equiv j.$$

In this way, space B_1 will be generated by the span ((1, 0, 0), (0, 1, 0), (0, 0, 1)) and space B_2 will be generated by the span $\{\text{id}, i, j\}$, such that the linear independence of both bases is inherited from the linear independence of the canonical base in R^3.

Since R^3 is a differentiable manifold, B-spaces are a differentiable manifold. However, these spaces are subject to and limited by the topology induced by the Euclidean metric. In this work, we carry out a step-by-step analysis of the properties of B-spaces to define a differentiable manifold not limited by the Euclidean metric, but at the same time influenced and defined by a particular property of Bermejo Algebras, the appearance of a "norm" in the real component of the double conjugated square of an element. We cannot define the squared norm of a treon in the conventional sense without introducing an inner product or a metric in the vector space. Here we present a pioneering analysis that allows working with norms, metrics, and inner products in a non-normed, non-metric space without inner products; at least not from the conventional viewpoint. This is because the products of Bermejo Algebras naturally produce these properties.

3. Construction of a Topology on Treons

3.1 Definitions for Treonic Topology

To construct a topology T on the space of treons, B2, which we denote X for simplicity, we define:

Let X be the set of treonic elements $p \equiv p_1\text{id} + p_2 i + p_3 j$, with $i^2 = j^2 = -\text{id}$, where id is the identity element of the algebra. To simplify, we will assume from here that id = 1. Let $P(X)$ be the power set of X, and let a topology $T \subseteq P(X)$. By definition, T must satisfy [6, 7]:

i. $\emptyset, X \in T$, where $\emptyset$ represents the empty set.
ii. If $U_1, U_2 \in T$, then $U_1 \cap U_2 \in T$ (finite intersection), where $U \in T$ are the open sets of topology T.
iii. If $(U_i) =_{i\in I} \in T$, then $U_{i\in I}\, U_i \in T$. The index I is an arbitrary index, finite or not.

By definition, we have two trivial topologies: the *indiscrete treonic topology*, $T = \{\emptyset, X\}$, and the *discrete treonic topology, $T = P(X)$.*

Let the pair (X, T) be our *treonic topological space*, we say that V is an open neighborhood of a treon

p if: $V \in T \wedge p \in V$.

3.2 Convergence of Treons and Definition of Hausdorff Space

We define a sequence of treons as a mapping φ such that:

$$\varphi : \mathbb{N} \to X,$$
$$n \mapsto \varphi(n) \equiv p_{\{n\}},$$

where $\{n\}$ is a notation to represent the successive elements:

$$p_{\{n\}} = p_{1\{1\}} + p_{2\{1\}}i + p_{3\{1\}}j, \quad p_{1\{2\}} + p_{2\{2\}}i + p_{3\{2\}}j, \quad p_{1\{3\}} + p_{2\{3\}}i + p_{3\{3\}}j, \ldots$$

We say that a sequence of treons $p\{n\}$ converges to an element p, and we denote $p\{n\} \to p$, if and only if:

$$\forall V, p \in V, \exists N \in \mathbb{N} : \forall n \geq N \implies p_{\{n\}} \in V.$$

The topological space (X, T) is called a Hausdorff space if and only if [3, 4]:

$$\forall p_a, p_b \in X, p_a \neq p_b, \exists V_{p_a} \in T \;\wedge\; V_{p_b} \in T : V_{p_a} \cap V_{p_b} = \emptyset,$$

where V_{pa} is the open neighborhood of p_a, and V_{pb} is an open neighborhood of p_b.

Since R^3 is a Hausdorff space with the topology induced by the Euclidean metric [3, 4], both space B_1 and space B_2 are Hausdorff spaces.

A Hausdorff space guarantees that the limits of sequences of points are unique [3, 4]. If a sequence $p\{n\}$ of treons converges to both p_a and p_b, necessarily $p_a = p_b$. In a Hausdorff space, a sequence that converges to two different points generates a contradiction.

If we have a Hausdorff space and if $p\{n\}$ converges to both p_a and p_b we have:

$$p_{\{n\}} \to p_a \Leftrightarrow \forall V_a,\, p_a \in V_a,\, \exists N_a \in \mathbb{N} : \forall n \geq N_a \Rightarrow p_{\{n\}} \in V_a,$$
$$p_{\{n\}} \to p_b \Leftrightarrow \forall V_b,\, p_b \in V_b,\, \exists N_b \in \mathbb{N} : \forall n \geq N_b \Rightarrow p_{\{n\}} \in V_b.$$

Therefore, taking simultaneously the maximum (N_a, N_b), we will have that $p_{\{n\}} \in V_a$ and $p\{n\} \in V_b$. This contradicts the definition of a Hausdorff space, where there is no point $p_{\{n\}}$ in the intersection $V_a \cap V_b$.

Hausdorff spaces are a fundamental pillar in the definition of manifolds [3, 4, 8, 9]; therefore, in this work, we develop the elementary mathematical tools for defining Hausdorff spaces under the analysis of Bermejo Algebras. This is the first step in generating treonic manifolds.

3.3 Definition of Treonic Quotient Topology

Let (X, T) be our treonic topological space, and let $\sim$ be an arbitrary equivalence relation on X; let $[p]\sim$ be an equivalence class for $p_i \in X$ under the relation $\sim$, and let $X/\sim$ be the quotient set. We define the mapping (surjective) canonical projection on the set of treons X as:

$$q : X \to X/\sim,$$
$$p \mapsto [p]_\sim.$$

Let an arbitrary Preimq$(U) \subseteq X$, such that Preimq$(U) \in T$, the canonical projection mapping q implies, by definition, the obligatory existence of an open set $U \subseteq X/\sim$, such that $U \in T^\sim$, where $T^\sim$ is a new topology defined on $X/\sim$, which is simply a quotient topology [10]for the case of treons. We call the pair $(X/\sim, T^\sim)$ the treons quotient space.

3.4 Analysis of the Real Component of $P \odot P$ $((*i,j))$

In algebra B, we define $\langle p^2 \rangle \equiv p \odot p$ $((*i,j))$, where $\odot$ is the product of algebra B, and $(*i,j)$ is the double complex conjugation of a treon, i.e., a conjugation in i and a conjugation in j [1, 2]. According to this, $\langle p^2\rangle = (\|p\|^2, 2p_1p_2 + p_2p_3, 2p_1p_3 + p_3p_2)$, where:

$$\|p\|^2 \equiv p_1^2 + p_2^2 + p_3^2.$$

The operation $\langle p^2 \rangle \equiv p \odot p$ $((*i,j))$ we name it "double conjugated square". Bermejo called $\|p\|^2$ the "norm squared", but it should not be interpreted as the norm squared per se of p, since only the real part of $\langle p^2 \rangle \in X$ is being considered. Note that this "norm squared" is the real part of the double conjugated square of a treon, i.e.,

$$\|p\|^2 \equiv \mathrm{Re}(\langle p^2\rangle).$$

We define the subset $N \subset X$, such that:

$$N \equiv \{p \in X : p = \langle p^2 \rangle\},$$

and the mapping $\langle\cdot\rangle$:

$$\langle \cdot^2 \rangle : X \to N \subset X,$$
$$p \mapsto \langle p^2 \rangle.$$

$\|p\|^2 \equiv \mathrm{Re}(\langle p^2\rangle)$ implies that the real part of a treon $\langle p^2 \rangle \in N$ is a place where Euclidean norms of three components of a real vector $\vec{p} \equiv p_1\hat{i} + p_2\hat{j} + p_3\hat{k}$ arise, whose components coincide with the components of the treon $p = p_1 + p_2 i + p_3 j$, an element of the domain of $\langle\cdot 2\rangle$. We understand that the norm of a real vector arises in the real component of a treon when performing the double conjugated square.

Note that $\|p\|^2 = 0 \Rightarrow p = 0 \wedge \langle p^2 \rangle = 0$. Therefore, within the set where norms are defined (the set N), any treon $\langle p^2 \rangle$ with a zero real part will be a zero treon, and its preimage under $\langle\cdot 2\rangle$ will be the zero treon in X. Therefore, in N, any pure imaginary treon $p_2 i + p_3 j$ is zero and derives from a zero treon; this means that there are no pure imaginary treons in N. This does not occur in $X \setminus N$, where a pure imaginary treon produces an element $\langle p_2 \rangle = (p^2 + p^2, p_2p_3, p_3p_2) \in N$.

Theorem: In N, the norm of a treon is zero if and only if the treon is zero:

$$\forall p \in N \subset X : \mathrm{Re}(p) = 0 \Leftrightarrow p = 0 + 0i + 0j.$$

Proof ⇒:

We have that $\langle p^2\rangle = (\|p\|^2, 2p_1p_2 + p_2p_3, 2p_1p_3 + p_3p_2)$, such that $\|p\|^2 \equiv p_1^2 + p_2^2 + p_3^2$. If $\|p\|^2 = 0 \Leftrightarrow p_1 = 0 \wedge p_2 = 0 \wedge p_3 = 0$, then $p = p_1 + p_2 i + p_3 j = 0 + 0i + 0j$.

Proof ⇐:

If $p = 0 + 0i + 0j$, then $p_1 = 0$. Therefore, $\mathrm{Re}(p) = 0$.
Taking into account the product $\odot$ of algebra B [1], defined as:

$$p_A \odot p_B \equiv (p_{A1}p_{B1} - p_{A2}p_{B2} - p_{A3}p_{B3}, p_{A1}p_{B2} + p_{A2}p_{B1} + p_{A3}p_{B2}, p_{A1}p_{B3} + p_{A3}p_{B1} + p_{A3}p_{B2}),$$

we have:

$$p_A \odot p_B^{((*_{i,j}))} = (p_{A1}p_{B1} + p_{A2}p_{B2} + p_{A3}p_{B3}, -p_{A1}p_{B2} + p_{A2}p_{B1} - p_{A3}p_{B2}, -p_{A1}p_{B3} + p_{A3}p_{B1} - p_{A3}p_{B2}),$$

where we will define $\mathrm{Re}(p_A \odot p_B^{((*_{i,j}))}) \equiv \mathrm{Re}(\langle p_A, p_B\rangle) \equiv p_A \diamond p_B \in \mathbb{R}$, which, while it has the structure of an inner product, we do not define it as such. The diamond operation $\diamond$ naturally arises from the definition of the product of algebra B. Therefore, we have:

$$\langle p_A, p_B\rangle = (p_A \diamond p_B, p_{A2}p_{B1} - p_{A1}p_{B2} - p_{A3}p_{B2}, p_{A3}p_{B1} - p_{A1}p_{B3} - p_{A3}p_{B2}).$$

Then, with this, we have an operation involving the product of a treon with another doubly conjugated treon, resulting in treons with a real component exhibiting the structure of an "inner product". We refer to the operation $\langle p_A, p_B\rangle$ as the *Bermejian inner product*. In this context, the case of the doubly conjugated square is considered a particular instance of the Bermejian inner product, specifically for the product of identical treons.

On the other hand, $\|p\|^2 = 0 \Rightarrow \vec{p} = \vec{0}$. Therefore, within N, any zero treon maps to the zero vector $\vec{p} = \vec{0}$ in $\mathbb{R}^3$ through a mapping h:

$$h : N \to \mathbb{R}^3,$$
$$\langle p^2\rangle \mapsto \vec{p}.$$

Taking the set of all treons $p \in X$ that have an image in $N \subset X$ under $\langle \cdot^2\rangle$, $\mathrm{Preim}_{\langle \cdot 2\rangle}$, we can define a composition mapping $H = h \circ \langle \cdot^2\rangle$, such that:

$$H : \mathrm{Preim}_{\langle \cdot 2\rangle} \to \mathbb{R}^3,$$

$$p \mapsto \vec{p}.$$

This mapping ensures that all elements have a defined norm, such that the only possibility for an element to have a zero norm is for the element to be zero (0, 0, 0).

Note that the components of $\vec{p}$ in the canonical basis, as defined, can be made to coincide with the components of a treon $p \in \mathrm{Preim}\ \langle \cdot 2\ \rangle$ in its corresponding base $\{id, i, j\}$. In this way:

$$\forall \vec{p} \in \mathbb{R}^3, \forall p \in \mathrm{Preim}_{\langle \cdot 2\rangle}, \exists \|\vec{p}\| \in \mathbb{R} : \|\vec{p}\| = \sqrt{\mathrm{Re}(\langle p^2\rangle)}.$$

The vectors $\vec{p} \in \mathbb{R}^3$ that are a representation of the treons $p \in \mathrm{Preim}_{\langle \cdot 2\rangle}$ under the composition mapping H, we denote as ρ.

3.5 S_r^2 -Spheres and Treonic Equivalence Classes

We define an S^2-sphere as the set of vectors ρ, such that $\mathrm{Re}\ (\langle p^2\rangle i) = r^2 \in R$:

$$S_i^2 \equiv \{\rho \in \mathbb{R}^3 : \sqrt{\mathrm{Re}(\langle p^2\rangle_i)} = r_i \wedge r_i > 0\},$$

where S_i^2 are the different spheres of radius r_i greater than zero. This notation, in the traditional sense, refers to 2-spheres of radius r. For the purposes of our algebra, we have renamed this as S^2 r-spheres.

With this, we construct an equivalence relation for the elements $p \in \mathrm{Preim}\ \langle \cdot 2\ \rangle$. Let the following equivalence relation $\sim$:

$$\forall p_i, p_j \in \mathrm{Preim}_{\langle \cdot 2\rangle} \subset X, \forall \rho_i, \rho_j \in \mathbb{R}^3 : p_i \sim p_j \Leftrightarrow \rho_i = \rho_j \ \veebar\ \rho_i = -\rho_j,$$

where $\veebar$ denotes the binary logical operator "exclusive or", which excludes the truth value "True" when both propositions are True.

Proof that $p_i \sim p_j$ is an equivalence relation:

3.5.1 Reflexivity: We have $p_i \sim p_i \Leftrightarrow \rho_i = \rho_i \veebar \rho_i = -\rho_i$. Clearly, this proposition is true because $\rho_i = 0$ imply null norms and, therefore, null vectors and treons that do not conform to the sphere by definition.

3.5.2 Symmetry: $p_i \sim p_j = p_j \sim p_i$. Clearly, this holds since $\rho_i = \rho_j$ is the same as $\rho_j = \rho_i$, and since $\rho_i = -\rho_j$ is the same as $\rho_j = -\rho_i$.

3.5.3 Transitivity: For $p_i \sim p_k$ and $p_k \sim p_j$ we have:

$$p_i \sim p_k \Leftrightarrow \rho_i = \rho_k \veebar \rho_i = -\rho_k,$$

$$p_k \sim p_j \Leftrightarrow \rho_k = \rho_j \veebar \rho_k = -\rho_j,$$

As $\rho_i = \rho_k = \rho_j$, and as $\rho_i = -\rho_k = -(-\rho_j) = \rho_j$, we have that the proposition $p_i \sim p_j$ is true as it verifies the unique possibility of transitivity that $\rho_i = \rho_j$. Thus, transitivity verifies $p_i \sim p_j \Leftrightarrow \rho_i = \rho_j$ only, being false $\rho_i = -\rho_j$.

We extended the definition of S_r^2-spheres to the treonic set X. We define:

$$\Lambda \equiv \{p \in \text{Preim}_{\langle \cdot^2 \rangle} \subset X : \sqrt{\text{Re}(\langle p^2 \rangle_i)} = r_i \wedge r_i > 0\}.$$

This is simply a change in the way of representing: ρ is a vector on the canonical base while p is a treon on the treonic base $\{\text{id}, i, j\}$.

We call the set Λ the "r-treosphere," again, a change in the way of thinking about the spaces where they are defined. For $r = 1$ we have a 1-treosphere such that there is an equivalence class $[p]_{\sim_{(r=1)}}$, for $r = 2$ we have $[p]_{\sim_{(r=2)}}$, and for any $r = r_0 \in \mathbb{R}$, we have $[p]_{\sim_{\mathbb{R}}}$. The set $[p]_{\sim_{\mathbb{R}}}$ for any $r_0 \in \mathbb{R}$ allows grouping the treons according to these equivalence classes.

We define $\Lambda/\sim$ as the set of all $[p]_{\sim_{\mathbb{R}}}$ in X. And we construct the *treons quotient space* $(\Lambda/\sim, \tilde{T})$. Note that $[p]_{\sim_{\mathbb{R}}}$ involves all the points in the volume of a sphere.

We denoted the topology T of (X, T) that is in Λ as T_Λ, and the corresponding topological subspace as (Λ, T_Λ). Thus, (Λ, T_Λ) is a topological subspace of (X, T).

4. The Treons Quotient Space is a Hausdorff Space

Let two treons p_i and $p_j \in \Lambda$, such that $\rho_i \not\models \rho_j \wedge \rho_i \not\models -\rho_j$, then $[pi]\sim \not\models [p_j]\sim$. Two treons pi and pj are equal if their corresponding components are equal; therefore, if $\rho_i \not\models \rho_j$ and $\rho_i \not\models -\rho_j$, then $p_i \not\models p_j$. On the other hand, the canonical projection mapping $q \circ h \circ \langle \cdot^2 \rangle : \Lambda \to \Lambda/\sim$ is surjective, not injective, and we can have two treons p_i and p_j mapped to an equivalence class $[p]\sim$, i.e., we can have two treons p_i and p_j such that $\rho_i = \rho_j \veebar \rho_i = -\rho_j$. This implies that the construction of a neighborhood in $\Lambda/\sim$ implies the definition of two neighborhoods in S_2 and, by extension, in Λ.

We define the following treons:

$$p_{(\zeta, r_0)} \equiv \{p_\zeta \in \Lambda : \rho_\zeta \Rightarrow \sqrt{\text{Re}(\langle p_\zeta^2 \rangle)} = r_0 > 0\},$$

$$p_{(-\zeta, r_0)} \equiv \{p_{(-\zeta)} \in \Lambda : \rho_{(-\zeta)} = -\rho_\zeta\},$$

where $p_{(\zeta, r_0)}$ and $p_{(-\zeta, r_0)}$ are the two elements of an equivalence class $[p_\zeta]_{\sim_{(r=r_0)}} \in \Lambda/\sim$.

Thus, we have:

$$H : \Lambda \to \mathbb{R}^3,$$

$$p_{(\zeta, r_0)} \mapsto \rho_{(\zeta, r_0)},$$

$$p_{(-\zeta, r_0)} \mapsto \rho_{(-\zeta, r_0)}.$$

Let the topological space (Λ, T_Λ), we can take a $p_{(\zeta,r_0)} \in V_{(\zeta,r_0)}$ such that $V_{(\zeta,r_0)} \in T_\Lambda$; therefore, $V_{(\zeta,r_0)}$ is an open neighborhood in (Λ, T_Λ). This can be done for any r and for any pair $p_\zeta, p_{(-\zeta)} \in \Lambda$. By definition of topology, $V_{(\zeta,r)} \cup V_{(-\zeta,r)} \in T_\Lambda$.

Additionally, let the topological spaces (Λ, T_Λ) and $(\Lambda/\sim, \tilde{T}_\Lambda)$, a set $V \subset \Lambda/\sim$ is said to be open in $\tilde{T}_\Lambda$ if and only if $\text{Preim}_q(V) \in T_\Lambda$, which is precisely the definition of *continuity* between topological spaces. Therefore, $\text{Preim}_q(V_\zeta) = V_{(\zeta,r)} \cup V_{(-\zeta,r)}$ implies that the canonical projection q is continuous, i.e., it implies the existence of an open $V_\zeta \in \tilde{T}_\Lambda$. $\tilde{T}_\Lambda$ is the quotient topology of the quotient space $(\Lambda/\sim, \tilde{T}_\Lambda)$.

We can choose neighborhoods $V_{(\mu,r_0)} \in T_\Lambda, V_{(-\mu,r_0)} \in T_\Lambda, V_{(\eta,r_0)} \in T_\Lambda$, and $V_{(-\eta,r_0)} \in T_\Lambda$, as small as we want; therefore, we can assume the following intersections:

$$\begin{aligned}
V_{(\mu,r_0)} \cap V_{(\eta,r_0)} &= \emptyset, \\
V_{(-\mu,r_0)} \cap V_{(-\eta,r_0)} &= \emptyset, \\
V_{(-\mu,r_0)} \cap V_{(\eta,r_0)} &= \emptyset, \\
V_{(\mu,r_0)} \cap V_{(-\eta,r_0)} &= \emptyset, \\
V_{(\eta,r_0)} \cap V_{(-\eta,r_0)} &= \emptyset, \\
V_{(-\mu,r_0)} \cap V_{(\mu,r_0)} &= \emptyset.
\end{aligned}$$

Using $\text{Preim}_q(V_\zeta) = V_{(\zeta,r)} \cup V_{(-\zeta,r)}$, we have:

$$\text{Preim}_q(V_\eta) = V_{(\eta,r_0)} \cup V_{(-\eta,r_0)},$$

$$\text{Preim}_q(V_\mu) = V_{(\mu,r_0)} \cup V_{(-\mu,r_0)},$$

such that $\text{Preim}_q(V_\eta) \in T_\Lambda, \text{Preim}_q(V_\mu) \in T_\Lambda, V_\eta \in \tilde{T}_\Lambda$, and $V_\eta \in \tilde{T}_\Lambda$.

Evaluating $\text{Preim}_q(V_\eta \cap V_\mu)$, we have:

$$\begin{aligned}
\text{Preim}_q(V_\eta \cap V_\mu) &= \text{Preim}_q(V_\eta) \cap \text{Preim}_q(V_\mu) \\
&= (V_{(\eta,r_0)} \cup V_{(-\eta,r_0)}) \cap (V_{(\mu,r_0)} \cup V_{(-\mu,r_0)}) \\
&= ((V_{(\eta,r_0)} \cup V_{(-\eta,r_0)}) \cap V_{(\mu,r_0)}) \cup ((V_{(\eta,r_0)} \cup V_{(-\eta,r_0)}) \cap V_{(-\mu,r_0)}) \\
&= (V_{(\eta,r_0)} \cap V_{(\mu,r_0)}) \cup (V_{(-\eta,r_0)} \cap V_{(\mu,r_0)}) \cup (V_{(\eta,r_0)} \cap V_{(-\mu,r_0)}) \cup (V_{(-\eta,r_0)} \cap V_{(-\mu,r_0)}) \\
&= \emptyset.
\end{aligned}$$

Therefore, $\text{Preim}_q(V_\eta \cap V_\mu) = \emptyset$.

Since the mapping q is surjective, each element of the codomain has at least one preimage in the domain. If we start from the premise $\text{Preim}_q(V_\eta \cap V_\mu) = \emptyset$, this implies that there is no element in Λ such that $q(p \in \Lambda) \in V_\eta \cap V_\mu$, i.e., there is no element in the domain of q that is mapped to the intersection $V_\eta \cap V_\mu$. Consequently, due to surjectivity, for there to be no $p \in \Lambda$ such that $q(p \in \Lambda) \in V\eta \cap V\mu$, it is necessary that there be no element in the intersection itself. If there were any element $[p]\sim \in V_\eta \cap V_\mu$, surjectivity guarantees that there necessarily exists some $p \in \Lambda$. Therefore:

$$\forall \, \text{Preim}_q(V_\eta \cap V_\mu) = \emptyset \Rightarrow V_\eta \cap V_\mu = \emptyset.$$

Let the topological space $(\Lambda/\sim, \tilde{T}_\Lambda)$:

$$\forall \, [p_\mu]_\sim, [p_\eta]_\sim \in \Lambda/\sim, \; [p_\mu]_\sim \neq [p_\eta]_\sim, \; \exists V_\mu \in \tilde{T}_\Lambda \;\wedge\; V_\eta \in \tilde{T}_\Lambda : V_\eta \cap V_\mu = \emptyset.$$

Therefore, the treon quotient topology $\Lambda/\sim$ is a Hausdorff space.

All this analysis allows us to well-define the treon quotient topology $\Lambda/\sim$ as a Hausdorff space, without the need to equip the vector space with norms or inner products. The definition of the product in Bermejo Algebras is sufficient to work with these concepts implicitly.

We must consider that we can also define a canonical projection mapping $m : \Lambda \to \Lambda/\sim$ that is not given by the composition $q \circ H$, and therefore does not depend on conventional vectors $\vec{p}$; however, this excludes from our analysis the equivalence classes of opposite vectors in the S^2 r-spheres and returns surface areas of radius r_i. Let us consider the mapping m:

$$m : \Lambda \to \Lambda / \sim$$

$$p \mapsto [p]_{\sim}$$

where:

$$\forall p_i, p_j \in \Lambda : p_i \sim p_j \iff \sqrt{\mathrm{Re}(\langle p_i^2 \rangle)} = \sqrt{\mathrm{Re}(\langle p_j^2 \rangle)}.$$

Note that now the equivalence classes are not two points represented by position vectors on S_2 but are each of the surfaces S_2. Similarly, we can now define other equivalence classes in their corresponding $\Lambda/\sim$, for example, taking the diamond $\diamond$ operation:

$$\forall p_i, p_j, p_k, p_l \in \Lambda : (p_i, p_j) \sim (p_k, p_l) \iff p_i \diamond p_j = p_k \diamond p_l.$$

Our work thus opens new possibilities in the development and topological analysis of Bermejo Alge- bras.

5. Conclusions

We conducted a topological analysis of the Bermejo Algebras, demonstrating their ability to create Hausdorff spaces without the need for traditional norms, inner products, or metrics. We achieved this distinctive feature through the creation of equivalence classes and a specific quotient topology. This contribution is particularly relevant for the study of complex functions and complex analysis in a non- associative algebraic framework.

We defined a Bermejian inner product derived from the product of the Bermejo Algebras; this product is distinguished by not depending on conventional definitions where vector spaces are equipped with metric, inner product, and norm operations. This Bermejian inner product naturally arose from the product of the Bermejo Algebras, allowing us to work with metric and norm properties without explicitly introducing these structures.

Our work established an important precedent for the exploration of new algebraic structures, topo- logical structures, and differential manifolds. By constructing treonic spheres and treonic equivalence classes, we provided a new way to group and analyze treons. Demonstrating that the treonic quotient space is a Hausdorff space ensured the uniqueness of the limits of point sequences in these spaces, which is fundamental for the definition and study of differential manifolds.

This work set an important precedent for the exploration of new algebraic and topological structures, opening a field of research for mathematicians and physicists.

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