ISSN: 2834-7706

Research Article

Journal of Mathematical Techniques and Computational Mathematics

First Exploration of a Novel Generalization of Lie and Malcev Algebras Leading to the Emergence of Complex Structures

Alejandro Jesus Bermejo Valdes*

Riojan Health Service, Piqueras 98, 26006, Logron˜o, La Rioja, Spain

*Corresponding Author
Alejandro Jesus Bermejo Valdes, Riojan Health Service, Piqueras 98, 26006, Logrono, La Rioja, Spain.

Submitted: 2024, Jul 01; **Accepted:** 2024, Jul 22; **Published:** 2024, Oct 16

Citation: Valdes, A. J. B. (2024). First Exploration of a Novel Generalization of Lie and Malcev Algebras Leading to the Emergence of Complex Structures. *J Math Techniques Comput Math, 3*(10), 01-04.

Abstract
We introduce a novel algebraic structure, termed algebra B, which generalizes Lie algebras through the definition of a non-associative and unital algebra. We demonstrate that algebra B retains essential properties such as bilinearity and antisymmetry, and satisfies both the Jacobi and Malcev identities; this occurs when the product operations of Lie and Malcev algebras are derived from the product operation defined in our algebra. By examining the connections between algebra B, Malcev algebras, and Lie algebras, we establish that algebra B effectively generalizes these structures. Furthermore, we show that algebra B can generate complex entities with a novel structure distinct from those currently known. Our findings lay the groundwork for future investigations into the practical applications and further theoretical development of this new algebraic framework.

Keywords: Algebra *B*, Lie Algebra, Malcev Algebra, Non-Associative Algebra

1. Introduction
Lie algebras are fundamental in both mathematics and theoretical physics due to their capacity to describe and analyze symmetries and continuous transformations. They find key applications in group theory, differential geometry, and representation theory, providing essential tools for studying algebraic and geometric structures. In theoretical physics, Lie algebras are crucial in quantum mechanics, quantum field theory, and general relativity, where they help understand the fundamental symmetries of physical laws and the behavior of particles and fields. This versatility and explanatory power make Lie algebras a central topic in advanced research [1,2].

The pursuit of generalizing mathematical definitions and exploring increasingly fundamental entities is a cornerstone of mathematical research. Lie algebras serve as a bridge between quantum mechanics and general relativity, motivating the search for a "unifying algebra" capable of integrating these two pivotal theories [2,3].

In this study, we present a comprehensive analysis of a novel algebraic structure, denoted as algebra B, designed to generalize Lie algebras through the definition of a non-associative and unital algebra. We demonstrate that the algebra B preserves essential algebraic properties such as bilinearity and antisymmetric, and satisfies both the Jacobi and Malcev identities when the products defining these algebras are derived from the product of the algebra B. By examining the interrelations between algebra B, Malcev algebras, and Lie algebras, we establish that algebra B functions as a valid and broad generalization of these structures. Moreover, we reveal that algebra B possesses the remarkable ability to generate imaginary units and complex entities with unique structures, distinct from those currently known. These findings open new avenues for theoretical investigation and practical application, providing a framework for future research in algebraic symmetries and transformations.

1.1 Lie Algebra
A Lie algebra is an algebraic structure consisting of a vector space g over a field K, together with a binary operation called the Lie bracket, denoted by $[,] : g{\times}g \rightarrow g$, which satisfies the following properties [3-5]:
- Bilinearity: $\forall\lambda, \gamma \in K$; $\forall a, b, c \in g$: $[\lambda a + \gamma b, c] = \lambda[a, c] + \gamma[b, c]$ and $[c, \lambda a + \gamma b] = \lambda[c, a] + \gamma[c, b]$.
- Antisymmetry: $\forall a, b \in g$: $[a, b] = -[b, a]$. This property implies that $\forall a \in g$: $[a, a] = 0$.
- Jacobi Identity: $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$, $\forall a, b, c \in g$.

This structure is not necessarily associative, as the antisymmetric property and the Jacobi identity do not imply the associativity of the Lie bracket operation [3,4].

1.2 Malcev Algebras
Malcev algebras, which are a direct generalization of Lie

algebras, form an important algebraic structure. While every Lie algebra is inherently a Malcev algebra, the converse is not true; not all Malcev algebras are Lie algebras. The defining feature of Malcev algebras is their satisfaction of a specific identity analogous to the Jacobi identity, known as the Malcev identity [6,7].

A Malcev algebra is defined as an algebraic structure comprising a vector space M over a field K, equipped with a binary operation $[,] : M \times M \to M$. This operation adheres to the following properties [6,7]:

- Antisymmetry: $\forall a, b \in M : [a, b] = -[b, a]$.
- Malcev Identity: $\forall a, b, c \in M : [[a, b], [a, c]] = [[[a, b], c], a] + [[[b, c], a], a] + [[[c, a], a], b]$.

The Malcev identity serves as a generalization of the Jacobi identity found in Lie algebras. Specifically, within the context of a Lie algebra, the Malcev identity simplifies to the Jacobi identity, highlighting the foundational relationship between these two algebraic structures [6,8].

Therefore, an algebra that aims to generalize both Lie and Malcev algebras must, by definition, satisfy the bilinearity of the product, antisymmetric, the Jacobi identity, and the Malcev identity.

2. Construction of Algebra $(V, \odot)$

2.1 Algebraic Structure of the Vector Space V

Let the set V be a mathematical entity that satisfies the Zermelo-Fraenkel axioms and the Axiom of Choice [9,10].

Let the algebraic structure of the vector space be defined as V $\equiv (V^3, +, \bullet)$ over the field F, where + is the addition operation in V 3 and the operation • denotes scalar multiplication of the elements of V^3 with the elements of the field F. The elements of V, which we will call vectors, are defined as ordered 3-tuples of the form (v_1, v_2, v_3) such that $v_i \in F$. Thus, we have defined a vector space over a field F whose vectors contain entries in F [11,12]. The addition operation + is defined as:

$+ : V \times V \to V,$

$((a_1, a_2, a_3), (b_1, b_2, b_3)) \to (c_1, c_2, c_3),$

$(a_1, a_2, a_3) + (b_1, b_2, b_3) \equiv (a_1 + b_1, a_2 + b_2, a_3 + b_3).$

This operation, defined through elements of F, is associative and commutative. The identity element of the addition is $(0, 0, 0) \equiv id_+$, and the inverse elements correspond to the additive inverses in F [11]: $(a_1, a_2, a_3) + (-a_1, -a_2, -a_3) = id_+$. Thus, the substructure $(V^3, +)$ is an abelian group [11,13].

The scalar multiplication operation • is defined as:

$\bullet : F \times V \to V,$

$(\lambda, (a_1, a_2, a_3)) \text{'}\!\!\to \lambda \bullet (a_1, a_2, a_3),$

$\lambda \bullet (a_1, a_2, a_3) \equiv (\lambda \bullet a_1, \lambda \bullet a_2, \lambda \bullet a_3), \lambda \in F.$

This operation is associative, commutative, distributive with respect to vector addition, and distributive with respect to addition in F [11]. The identity element corresponds to the identity in F : $id_\bullet \bullet (a_1, a_2, a_3) = (a_1, a_2, a_3)$. Thus, the substructure $(V^3, \bullet)$ is a commutative monoid [11,13].

Therefore, the complete structure $V \equiv (V^3, +, \bullet)$ is a commutative ring [11,13].

2.2 Definition of the Algebra $(V, \odot)$

We equip the structure V with a bilinear product operation $\odot$, thus generating an F -algebra B that we denote as $B \equiv (V, \odot)$.

Let the arbitrary 3-tuples (a_1, a_2, a_3) and (b_1, b_2, b_3), we define the product operation: The operation
$\odot$ is defined as:

$\odot : V \times V \to V,$

$((a_1, a_2, a_3), (b_1, b_2, b_3)) \text{'}\!\!\to (c_1, c_2, c_3),$

$(a_1, a_2, a_3) \odot (b_1, b_2, b_3) \equiv$

$\equiv (a_1 \bullet b_1 - a_2 \bullet b_2 - a_3 \bullet b_3, a_1 \bullet b_2 + a_2 \bullet b_1 + a_3 \bullet b_2, \quad a_1 \bullet b_3 + a_3 \bullet b_2 + a_3 \bullet b_1).$

In algebra B, the equality between elements is defined as follows: $\forall a, b \in F : (a_1, a_2, a_3) = (b_1, b_2, b_3) \Leftrightarrow a_1 = b_1 \wedge a_2 = b_2 \wedge a_3 = b_3$

2.2.1 Bilinearity of the Product

Our definition of the product satisfies (see Appendix A) the following three properties of bilinearity:

- Left distributivity with respect to addition +: $(a_1, a_2, a_3) \odot ((b_1, b_2, b_3) + (c_1, c_2, c_3)) = (a_1, a_2, a_3) \odot (b_1, b_2, b_3) + (a_1, a_2, a_3) \odot (c_1, c_2, c_3)$.
- Right distributivity with respect to addition +: $((a_1, a_2, a_3) + (b_1, b_2, b_3)) \odot (c_1, c_2, c_3) = (a_1, a_2, a_3) \odot (c_1, c_2, c_3) + (b_1, b_2, b_3) \odot (c_1, c_2, c_3)$.
- Multiplication • of field elements with respect to $\odot$: $\forall \lambda \in F$: $(\lambda \bullet (a_1, a_2, a_3)) \odot (b_1, b_2, b_3) = (a_1, a_2, a_3) \odot (\lambda \bullet (b_1, b_2, b_3)) = \lambda \bullet ((a_1, a_2, a_3) \odot (b_1, b_2, b_3))$.

This last property arises directly from the commutativity of the entries of the vectors, which are elements of F, with respect to the elements of the field F of the algebra.

2.2.2 Basic Properties of Algebra B

Non-Associativity and Non-Alternativity

Our algebra exhibit non-associativity (see Appendix B), hence: $((a_1, a_2, a_3) \odot (b_1, b_2, b_3)) \odot (c_1, c_2, c_3) \neq (a_1, a_2, a_3) \odot ((b_1, b_2, b_3) \odot (c_1, c_2, c_3))$.

We define the Associator operator as ASSOC := $(X \odot Y) \odot Z - X \odot (Y \odot Z)$, such that $X, Y, Z \in V$. According to this definition:

ASSOC $= ((a_1, a_2, a_3) \odot (b_1, b_2, b_3)) \odot (c_1, c_2, c_3)$

$— (a_1, a_2, a_3) \odot ((b_1, b_2, b_3) \odot (c_1, c_2, c_3))$

$= (d, f, g),$

where: $d = -(a_3 \bullet b_2 \bullet c_2) - (a_3 \bullet b_2 \bullet c_3) + (a_2 \bullet b_3 \bullet c_2) + (a_3 \bullet b_3 \bullet c_2)$, $f = (-a_3 \bullet b_3 \bullet c_2) + (a_3 \bullet b_2 \bullet c_2) + (a_2 \bullet b_3 \bullet c_3) - (a_3 \bullet b_3 \bullet c_2)$, and $g = (-a_2 \bullet b_2 \bullet c_3) + (a_3 \bullet b_2 \bullet c_2) - (a_3 \bullet b_3 \bullet c_2) + (a_3 \bullet b_2 \bullet c_2)$. This demonstrates that our algebra is a non-associative F -algebra.

A non-associative algebra can be an alternative algebra if it satisfies $X(XY) = (XX)Y$ and $(Y X)X = Y (XX)$ for all X, Y in the algebra [14,15]. We examine this property because an alternative algebra may be made into a Malcev algebra by defining the Malcev product as $[X, Y] \equiv XY - Y X$ [6,8,14,15].

We explored both properties: $X(XY) = (XX)Y$ and $(Y X)X = Y (XX)$.

We verified that the property $X(XY) = (XX)Y$ does not hold in

our algebra (see Appendix *C*), and for this property, we defined an Alternativator operator, $ALTER_1$, such that:
$ALTER_1 \equiv X \odot (X \odot Y) - (X \odot X) \odot Y.$

Our Alternativator $ALTER_1$ resulted in the following:

$$ALTER_1 = -a_3 \bullet a_3 \bullet b_2 + a_3 \bullet a_2 \bullet b_3,$$
$$-a_2 \bullet a_3 \bullet b_3 + a_3 \bullet a_3 \bullet b_2 + a_3 \bullet a_3 \bullet b_2 - a_3 \bullet a_2 \bullet b_2,$$
$$a_3 \bullet a_3 \bullet b_2 - a_3 \bullet a_2 \bullet b_2 + a_2 \bullet a_2 \bullet b_3 - a_3 \bullet a_2 \bullet b_2 .$$

The property $(Y X)X = Y (XX)$ also did not hold in our algebra (see Appendix C). For this case, we defined the Alternativator operator, $ALTER_2$, as follows:
$ALTER_2 \equiv (Y \odot X) \odot Y - Y \odot (X \odot X) =$
$(-b_2{\bullet}a_3{\bullet}a_2{+}b_3{\bullet}a_2{\bullet}a_2,{-}b_2{\bullet}a_3{\bullet}a_3{+}b_3{\bullet}a_3{\bullet}a_2{+}b_3{\bullet}a_3{\bullet}a_2{-}b_3{\bullet}a_2{\bullet}a_2,$
$b_3{\bullet}a_3{\bullet}a_2{-}b_3{\bullet}a_2{\bullet}a_2{+}b_2{\bullet}a_2{\bullet}a_3{-}b_3{\bullet}a_2{\bullet}a_2).$

Therefore, the algebra *B* is non-associative and non-alternative.

2.2.3 Non-Commutativity and Antisymmetry

We define the Commutator (or Bracket) operator $[X, Y] \equiv X \odot Y - Y \odot X$, where $X, Y \in V$. According to this definition:
$[(a_1, a_2, a_3), (b_1, b_2, b_3)] =$
$(a_1, a_2, a_3) \odot (b_1, b_2, b_3) - (b_1, b_2, b_3) \odot (a_1, a_2, a_3) = (0, a_3 \bullet b_2 - b_3 \bullet a_2, a_3 \bullet b_2 - b_3 \bullet a_2) = \beta \bullet (0,1,1),$
where $\beta \equiv a_3 \bullet b_2 - b_3 \bullet a_2$, 1 represents the identity element for multiplication $\bullet$ in the field F, and 0 denotes the identity element for addition in F. This demonstrates that our algebra is non-commutative.
Note that the commutator results in the cancellation of the first component. Furthermore, it reveals the field element $\beta = a_3 \bullet b_2 - b_3 \bullet a_2$. In this way, our commutator has the structure of a determinant:

where $u \equiv (1, 0, 0)$, $a \equiv (a_1, a_2, a_3)$ and $b \equiv (b_1, b_2, b_3)$.

Therefore, we can define our commutator generically as:
$[(a_1, a_2, a_3), (b_1, b_2, b_3)] = -\det(u, a, b) \bullet (0, 1, 1),$

If we evaluate the commutator of the same element, we have:
$[(a_1, a_2, a_3), (a_1, a_2, a_3)] = (a_1, a_2, a_3) \odot (a_1, a_2, a_3) - (a_1, a_2, a_3) \odot (a_1, a_2, a_3) = (0, 0, 0).$
On the other hand, let's analyze the following commutators:

Thus:

$[(a_1, a_2, a_3), (b_1, b_2, b_3)] = (0, a_3 \bullet b_2 - b_3 \bullet a_2, a_3 \bullet b_2 - b_3 \bullet a_2),$
$[(b_1, b_2, b_3), (a_1, a_2, a_3)] = -(0, a_3 \bullet b_2 - a_2 \bullet b_3, a_3 \bullet b_2 - b_3 \bullet a_2) = -[(a_1, a_2, a_3), (b_1, b_2, b_3)].$

$[(a_1, a_2, a_3), (b_1, b_2, b_3)] = -[(b_1, b_2, b_3), (a_1, a_2, a_3)],$

where this property manifests as the antisymmetry of the operation $[X, Y]$ when defined via the product $\odot$ in algebra *B*.
Note that algebra B itself is not antisymmetric, due to:
$(a_1, a_2, a_3) \odot (b_1, b_2, b_3) \neq -(b_1, b_2, b_3) \odot (a_1, a_2, a_3).$

2.2.4 Identity Element

The identity element of the product $\odot$ in *B* is (1, 0, 0). According to this, we have: $(a_1, a_2, a_3) \odot (1, 0, 0) = (1, 0, 0) \odot (a_1, a_2, a_3) = (a_1, a_2, a_3)$.
Knowing that $u \equiv (1, 0, 0)$, we can redefine the commutator as:
$[(a_1, a_2, a_3), (b_1, b_2, b_3)] = -\det(id_\odot, a, b) \bullet (0, 1, 1)$, where we have redefined u as $id_\odot \equiv (1, 0, 0)$.
The existence of an identity element makes our algebra a unital algebra.

2.3 Relation of Algebra *B* with Malcev and Lie Algebras

Our algebra is non-commutative and non-associative. Due to their structural properties, Malcev and Lie algebras could be related to our algebra. To verify the similarities between algebra B and both Malcev and Lie algebras, we need to demonstrate that algebra B satisfies the defining properties of Malcev and Lie algebras.

2.3.1 Relation of Algebra *B* with Lie Algebra

To demonstrate that a Lie algebra derives from our algebra B, it is necessary to show that the Lie bracket defined from our product $\odot$ satisfies the properties of bilinearity and the Jacobi identity, as anti- symmetry has been previously demonstrated in section "2.2.2. Non-Commutativity and Antisymmetry".

• Bilinearity of the Commutator

We have established that the bilinearity of the Lie bracket, defined from the product $\odot$, holds true (see Appendix D). This confirms that the following property is satisfied in our algebra:
$\forall \lambda, \gamma \in F : [\lambda(a_1, a_2, a_3) + \gamma(b_1, b_2, b_3), (c_1, c_2, c_3)] = \lambda[(a_1, a_2, a_3), (c_1, c_2, c_3)] + \gamma[(b_1, b_2, b_3), (c_1, c_2, c_3)].$

• Jacobi Identity

The Jacobi identity [3, 4] holds in algebra B when we define the Lie bracket through our product $\odot$. Therefore, the Jacobi identity is a property that represents a particular case in algebra *B* (see Appendix *E*).

2.3.2 Construction of Malcev Algebra through Algebra *B*

The Malcev identity is satisfied if $\forall a, b, c \in M : [[a, b], [a, c]] = [[[a, b], c], a] + [[[b, c], a], a] + [[[c, a], a], b]$ [6, 7].

We verify that the algebra B satisfies the Malcev identity by defining the brackets in terms of the product $\odot$ (see Appendix *F*). In fact, it satisfies the identity in a particular way; in our algebra, this identity has the following structure:
$\forall a, b, c \in V : [[a, b], [a, c]] = [[[a, b], c], a] + [[[b, c], a], a] + [[[c, a], a], b] = (0, 0, 0).$
Thus, for algebra *B*, the following identities hold:
$[[[a, b], c], a] = -[[[b, c], a], a] - [[[c, a], a], b],$
$[[[b, c], a], a] = -[[[a, b], c], a] - [[[c, a], a], b],$
$[[[c, a], a], b] = -[[[a, b], c], a] - [[[b, c], a], a].$

3. Generation of Complex Entities from *B*

We extracted other basic properties of the algebra *B* by evaluating the product of several vectors with particular structures. From this analysis, imaginary units emerged for our algebra. Thus, the algebra *B* is capable of generating complex entities.

3.1 Imaginary Unit i in *B*

Let the product $(0, a_2, 0) \odot (0, b_2, 0) = (-a_2 \bullet b_2, 0, 0)$. In this way, for the case $a_2 = b_2 = 1$, we have: $(0, 1, 0) \odot (0, 1, 0) = (-1, 0, 0) = -id_\odot$.
Which, if we define $i \equiv (0, 1, 0)$ and $i^2 \equiv i \odot i$ results in:
$i^2 = -id_\odot$.

Note also that $(-1, 0, 0) \odot (-1, 0, 0) = (1, 0, 0) = \mathrm{id}_{\odot}$.

3.2 Imaginary Unit *j* in *B*

Let the product $(0, 0, a_3) \odot (0, 0, b_3) = (-a_3 \cdot b_3, 0, 0)$. In this way, if $a_3 = b_3 = 1$, we have:
$(0, 0, 1) \odot (0, 0, 1) = -\mathrm{id}_{\odot}$.

Defining $j \equiv (0, 0, 1)$ results in:

$j^2 = -\mathrm{id}_{\odot}$.

We then define the complex number $b = (b_1, b_2, b_3)$ such that:
$b_1 = \mathrm{Re}(b)$,
$b_2 = \mathrm{Im}_1(b)$, $b_3 = \mathrm{Im}_2(b)$,
where Re(b) is read as "real part of b", $\mathrm{Im}_1(b)$ is read as "first imaginary part of b", and $\mathrm{Im}_2(b)$ is read as "second imaginary part of b".
Consistent with this, we may consider the elements (b1, b2, b3) of the algebra *B* as complex entities, such that the following expression can be defined:
$b = (b_1, b_2, b_3) = b_1 \cdot (1, 0, 0) + b_2 \cdot (0, 1, 0) + b_3 \cdot (0, 0, 1) = b_1 + b_2 \cdot i + b_3 \cdot j$, $i^2 = j^2 = -\mathrm{id}_{\odot}$.

4. Other Properties

4.1 Nullification of the Product a • *i* ⊙ *b* • *j*

In our algebra, the product of the form (0, a2, 0) □ (0, 0, b3) results in a null vector, 0 ≡ (0,0,0).
Therefore,
$a \cdot i \odot b \cdot j = 0$.
We will call this property "orthomulearity", simply because it resembles the orthogonality of vector spaces under the inner product. In our algebra, when two elements of the type a • i and *b* • j are multiplied with
⊙, the result nullifies regardless of the factors involved.

4.2 Product $a_3 \cdot j \odot b_2 \cdot i$

Let $(0, 0, a_3) \odot (0, b_2, 0)$, we have:
$(0, 0, a_3) \odot (0, b_2, 0) = (0, a_3 \cdot b_2, a_3 \cdot b_2) = a_3 \cdot b_2 \cdot (0, 1, 1)$,
and, for $a_3 = b_2 = 1$:
$(0, 0, 1) \odot (0, 1, 0) = (0, 1, 1)$.
Writing $(0, 0, a_3) \odot (0, b_2, 0)$ in the form $a_3 \cdot j \odot b_2 \cdot i$, we have:
$a_3 \cdot j \odot b_2 \cdot i = (a_3 \cdot b_2) \cdot j \odot i = (a_3 \cdot b_2) \cdot (0, 1, 1) = (a_3 \cdot b_2) \cdot ((0, 0, 1) \odot (0, 1, 0))$.

5. Conclusions

We have introduced a novel algebraic structure termed algebra *B,* which generalizes Lie algebras through the definition of a non-associative and unital algebra. Our results demonstrate that algebra *B* retains essential properties such as bilinearity and antisymmetry, and satisfies both the Jacobi and Malcev identities when we define the product of their corresponding algebras through the product of our algebra.

We have shown that algebra *B* can be considered a valid generalization of Malcev and Lie algebras, highlighting its potential to integrate fundamental algebraic structures. Additionally, the ability of algebra *B* to generate imaginary units and complex entities opens new avenues for research in algebraic symmetries and transformations.

Future research should focus on exploring its practical applications and extending the theoretical framework established here.

Our study significantly contributes to the field of algebra theory, offering a new perspective and potential tools for the investigation of algebraic structures and their applications across various scientific disciplines. This research sets the stage for future explorations in the quest for a unifying algebra that can integrate key concepts in mathematics and theoretical physics.

References

1. Iachello, F. (2006). *Lie algebras and applications* (Vol. 708, pp. xiv+-196). Berlin: Springer.
2. de Azcárraga, J. A., & Izquierdo, J. M. (1998). *Lie groups, Lie algebras, cohomology and some applications in physics*. Cambridge University Press.
3. Gilmore, R. (2006). *Lie groups, Lie algebras, and some of their applications*. Courier Corporation.
4. Serre, J. P. (2009). *Lie algebras and Lie groups: 1964 lectures given at Harvard University.* Springer.
5. Varadarajan, V. S. (2013). *Lie groups, Lie algebras, and their representations* (Vol. 102). Springer Science & Business Media.
6. Sagle, A. A. (1961). Malcev algebras. *Transactions of the American Mathematical Society, 101*(3), 426-458.
7. Nagy, P. T. (1993). Moufang loops and Malcev algebras. Sem. *Sophus Lie, 3*(1), 65-68.
8. Madariaga, S. (2017). Splitting of operations for alternative and Malcev structures. *Communications in Algebra, 45*(1), 183-197.
9. Rathjen, M. (2012). Constructive Zermelo-Fraenkel set theory, power set, and the calculus of constructions. In Epistemology versus Ontology: Essays on the Philosophy and Foundations of Mathematics in Honour of Per Martin-Löf (pp. 313-349). Dordrecht: Springer Netherlands.
10. Jech, T. J. (1977). About the axiom of choice. In *Studies in Logic and the Foundations of Mathematics* (Vol. 90, pp. 345-370). Elsevier.
11. Spindler, K. (2018). Abstract Algebra with Applications: Volume 1: Vector Spaces and Groups. CRC Press.
12. Amitsur, A. S. (1956). Algebras over infinite fields. *Proceedings of the American Mathematical Society, 7*(1), 35-48.
13. Passman, D. S. (2011). *The algebraic structure of group rings.* Courier Corporation.
14. Schafer, R. D. (1943). Alternative algebras over an arbitrary field.
15. Elduque, A., Myung, H. C., Elduque, A., & Myung, H. C. (1994). *Mutations of alternative algebras* (pp. 92-135). Springer Netherlands.

A. Proof of the bilinearity of the product $\odot$
Left distributivity with respect to Addition +

$(a_1, a_2, a_3) \odot ((b_1, b_2, b_3) + (c_1, c_2, c_3)) = (a_1, a_2, a_3) \odot (b_1, b_2, b_3) + (a_1, a_2, a_3) \odot (c_1, c_2, c_3).$

Left-hand side: Expanding +:

$$\begin{aligned}(a_1, a_2, a_3) \odot ((b_1, b_2, b_3) + (c_1, c_2, c_3)) &= (a_1, a_2, a_3) \odot (b_1 + c_1, b_2 + c_2, b_3 + c_3)\\ &= (a_1 \cdot (b_1 + c_1) - a2 \cdot (b_2 + c_2) - a3 \cdot (b3 + c3),\\ &\quad a_1 \cdot (b_2 + c_2) + a_2 \cdot (b_1 + c_1) + a3 \cdot (b_2 + c_2),\\ &\quad a_1 \cdot (b3 + c3) + a_3 \cdot (b_2 + c_2) + a_3 \cdot (b_1 + c_1))\\ &= (a_1 \cdot b_1 + a_1 \cdot c_1 - a_2 \cdot b_2 - a_2 \cdot c_2 - a_3 \cdot b_3 - a_3 \cdot c_3,\\ &\quad a_1 \cdot b_2 + a_1 \cdot c_2 + a_2 \cdot b_1 + a_2 \cdot c_1 + a_3 \cdot b_2 + a_3 \cdot c_2,\\ &\quad a_1 \cdot b_3 + a_1 \cdot c_3 + a_3 \cdot b_2 + a_3 \cdot c_2 + a_3 \cdot b_1 + a_3 \cdot c_1).\end{aligned}$$

Right-Hand Side:

$$\begin{aligned}(a_1, a_2, a_3) \odot (b_1, b_2, b_3) + (a_1, a_2, a_3) \odot (c_1, c_2, c_3) &= (a_1 \cdot b_1 - a_2 \cdot b_2 - a_3 \cdot b_3 + a_1 \cdot c_1 - a_2 \cdot c_2 - a_3 \cdot c_3,\\ &\quad a_1 \cdot b_2 + a_2 \cdot b_1 + a_3 \cdot b_2 + a_1 \cdot c_2 + a_2 \cdot c_1 + a_3 \cdot c_2,\\ &\quad a_1 \cdot b_3 + a_3 \cdot b_2 + a_3 \cdot b_1 + a_1 \cdot c_3 + a_3 \cdot c_2 + a_3 \cdot c_1).\end{aligned}$$

Left-Hand Side = Right-Hand Side.

Right Distributivity with respect to Addition +

$((a_1, a_2, a_3) + (b_1, b_2, b_3)) \odot (c_1, c_2, c_3) = (a_1, a_2, a_3) \odot (c_1, c_2, c3) + (b_1, b_2, b3) \odot (c_1, c_2, c_3).$

Left-Hand Side:

$$\begin{aligned}((a_1, a_2, a_3) + (b_1, b_2, b_3)) \odot (c_1, c_2, c_3) &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \odot (c_1, c_2, c_3)\\ &= ((a_1 + b_1) \cdot c_1 - (a_2 + b_2) \cdot c_2 - (a_3 + b_3) \cdot c_3,\\ &\quad (a_1 + b_1) \cdot c_2 + (a_2 + b_2) \cdot c_1 + (a_3 + b_3) \cdot c_2,\\ &\quad (a_1 + b_1) \cdot c_3 + (a_3 + b_3) \cdot c_2 + (a_3 + b_3) \cdot c_1\\ &= (a_1 \cdot c_1 + b_1 \cdot c_1 - a_2 \cdot c_2 - b_2 \cdot c_2 - a_3 \cdot c_3 - b_3 \cdot c_3,\\ &\quad a_1 \cdot c_2 + b_1 \cdot c_2 + a_2 \cdot c_1 + b_2 \cdot c_1 + a_3 \cdot c_2 + b_3 \cdot c_2,\\ &\quad a_1 \cdot c3 + b_1 \cdot c_3 + a_3 \cdot c_2 + b_3 \cdot c_2 + a_3 \cdot c_1 + b_3 \cdot c_1).\end{aligned}$$

Right-Hand Side:

$$\begin{aligned}(a_1, a_2, a_3) \odot (c_1, c_2, c_3) + (b_1, b_2, b_3) \odot (c_1, c_2, c_3) &= a_1 \cdot c_1 - a_2 \cdot c_2 - a_3 \cdot c_3,\\ &\quad a_1 \cdot c_2 + a_2 \cdot c_1 + a_3 \cdot c_2,\end{aligned}$$

$$a_1 \cdot c_3 + a_3 \cdot c_2 + a_3 \cdot c_1) + (b_1 \cdot c_1 - b_2 \cdot c_2 - b_3 \cdot c_3,$$
$$b_1 \cdot c_2 + b_2 \cdot c_1 + b_3 \cdot c_2,$$
$$b_1 \cdot c_3 + b_3 \cdot c_2 + b_3 \cdot c_1$$
$$= (a_1 \cdot c_1 - a_2 \cdot c_2 - a_3 \cdot c_3 + b_1 \cdot c_1 - b_2 \cdot c_2 - b_3 \cdot c_3,$$
$$a_1 \cdot c_2 + a_2 \cdot c_1 + a_3 \cdot c_2 + b_1 \cdot c_2 + b_2 \cdot c_1 + b_3 \cdot c_2,$$
$$a_1 \cdot c_3 + a_3 \cdot c_2 + a_3 \cdot c_1 + b_1 \cdot c_3 + b_3 \cdot c_2 + b_3 \cdot c_1.$$

Left-Hand Side = Right-Hand Side.

Multiplication $\cdot$ of field elements with respect to $\odot$

$(\lambda \cdot (a_1, a_2, a_3)) \odot (b_1, b_2, b_3) = (a_1, a_2, a_3) \odot (\lambda \cdot (b_1, b_2, b_3)) = \lambda \cdot ((a_1, a_2, a_3) \odot (b_1, b_2, b_3)), \lambda \in F.$

This last property arises directly from the commutativity that exists in the entries of the vectors, which are elements of the field F of the algebra.

B Proof of the Non-Associativity of the Product $\odot$

We initially evaluate:

$$((a_1, a_2, a_3) \odot (b_1, b_2, b_3)) \odot (c_1, c_2, c_3) = ((a_1 \cdot b_1 - a_2 \cdot b_2 - a_3 \cdot b_3,$$
$$a_1 \cdot b_2 + a_2 \cdot b_1 + a_3 \cdot b_2,$$
$$a_1 \cdot b_3 + a_3 \cdot b_2 + a_3 \cdot b_1) \odot (c_1, c_2, c_3))$$
$$= ((a_1 \cdot b_1 - a_2 \cdot b_2 - a_3 \cdot b_3) \cdot c_1$$
$$- (a_1 \cdot b_2 + a_2 \cdot b_1 + a_3 \cdot b_2) \cdot c_2$$
$$- (a_1 \cdot b_3 + a_3 \cdot b_2 + a_3 \cdot b_1) \cdot c_3,$$
$$(a_1 \cdot b_1 - a_2 \cdot b_2 - a_3 \cdot b_3) \cdot c_2$$
$$+ (a_1 \cdot b_2 + a_2 \cdot b_1 + a_3 \cdot b_2) \cdot c_1$$
$$+ (a_1 \cdot b_3 + a_3 \cdot b_2 + a_3 \cdot b_1) \cdot c_2,$$
$$(a_1 \cdot b_1 - a_2 \cdot b_2 - a_3 \cdot b_3) \cdot c_3$$
$$+ (a_1 \cdot b_3 + a_3 \cdot b_2 + a_3 \cdot b_1) \cdot c_2$$
$$+ (a_1 \cdot b_3 + a_3 \cdot b_2 + a_3 \cdot b_1) \cdot c_1).$$

Now we evaluate:

$$(a_1, a_2, a_3) \odot ((b_1, b_2, b_3) \odot (c_1, c_2, c_3)) = (a_1, a_2, a_3) \odot (b_1 \cdot c_1 - b_2 \cdot c_2 - b_3 \cdot c_3,$$
$$b_1 \cdot c_2 + b_2 \cdot c_1 + b_3 \cdot c_2,$$
$$b_1 \cdot c_3 + b_3 \cdot c_2 + b_3 \cdot c_1)$$
$$= a_1 \cdot (b_1 \cdot c_1 - b_2 \cdot c_2 - b_3 \cdot c_3)$$
$$- a_2 \cdot (b_1 \cdot c_2 + b_2 \cdot c_1 + b_3 \cdot c_2)$$
$$- a_3 \cdot (b_1 \cdot c_3 + b_3 \cdot c_2 + b_3 \cdot c_1), a_1 \cdot (b_1 \cdot c_2 + b_2 \cdot c_1 + b_3 \cdot c_2)$$

$$+a_2 \cdot (b_1 \cdot c_1 - b_2 \cdot c_2 - b_3 \cdot c_3)$$
$$+a_3 \cdot (b_1 \cdot c_2 + b_2 \cdot c_1 + b_3 \cdot c_2),$$
$$a_1 \cdot (b_1 \cdot c_3 + b_3 \cdot c_2 + b_3 \cdot c_1)$$
$$+a_3 \cdot (b_1 \cdot c_2 + b_2 \cdot c_1 + b_3 \cdot c_2)$$
$$+a_3 \cdot (b_1 \cdot c_1 - b_2 \cdot c_2 - b_3 \cdot c_3)\ .$$

Comparing component by component:

First Component:

Second Component:

$$(a_1 \cdot b_1 \cdot c_1 - a_2 \cdot b_2 \cdot c_1 - a_3 \cdot b_3 \cdot c_1)$$
$$—(a_1 \cdot b_2 \cdot c_2 + a_2 \cdot b_1 \cdot c_2 + a_3 \cdot b_2 \cdot c_2)$$
$$—(a_1 \cdot b_3 \cdot c_3 + a_3 \cdot b_2 \cdot c_3 + a_3 \cdot b_1 \cdot c_3)$$
$$\neq (a_1 \cdot b_1 \cdot c_1 - a_1 \cdot b_2 \cdot c_2 - a_1 \cdot b_3 \cdot c_3)$$
$$—(a_2 \cdot b_1 \cdot c_2 + a_2 \cdot b_2 \cdot c_1 + a_2 \cdot b_3 \cdot c_2)$$
$$—(a_3 \cdot b_1 \cdot c_3 + a_3 \cdot b_3 \cdot c_2 + a_3 \cdot b_3 \cdot c_1).$$
$$—(a_3 \cdot b_2 \cdot c_2) - (a_3 \cdot b_2 \cdot c_3)$$
$$\neq -(a_2 \cdot b_3 \cdot c_2) - (a_3 \cdot b_3 \cdot c_2).$$

$$(a_1 \cdot b_1 \cdot c_2 - a_2 \cdot b_2 \cdot c_2 - a_3 \cdot b_3 \cdot c_2)$$
$$+(a_1 \cdot b_2 \cdot c_1 + a_2 \cdot b_1 \cdot c_1 + a_3 \cdot b_2 \cdot c_1)$$
$$+(a_1 \cdot b_3 \cdot c_2 + a_3 \cdot b_2 \cdot c_2 + a_3 \cdot b_1 \cdot c_2)$$
$$\neq (a_1 \cdot b_1 \cdot c_2 + a_1 \cdot b_2 \cdot c_1 + a_1 \cdot b_3 \cdot c_2)$$
$$+(a_2 \cdot b_1 \cdot c_1 - a_2 \cdot b_2 \cdot c_2 - a_2 \cdot b_3 \cdot c_3)$$
$$+(a_3 \cdot b_1 \cdot c_2 + a_3 \cdot b_2 \cdot c_1 + a_3 \cdot b_3 \cdot c_2).$$

Third Component:
In this way:

$$(-a_3 \cdot b_3 \cdot c_2) + (a_3 \cdot b_2 \cdot c_2)$$
$$\neq (-a_2 \cdot b_3 \cdot c_3) + (a_3 \cdot b_3 \cdot c_2).$$

$$(a_1 \cdot b_1 \cdot c_3 - a_2 \cdot b_2 \cdot c_3 - a_3 \cdot b_3 \cdot c_3)$$
$$+(a_1 \cdot b_3 \cdot c_2 + a_3 \cdot b_2 \cdot c_2 + a_3 \cdot b_1 \cdot c_2)$$
$$+(a_1 \cdot b_3 \cdot c_1 + a_3 \cdot b_2 \cdot c_1 + a_3 \cdot b_1 \cdot c_1)$$
$$\neq (a_1 \cdot b_1 \cdot c_3 + a_1 \cdot b_3 \cdot c_2 + a_1 \cdot b_3 \cdot c_1)$$
$$+(a_3 \cdot b_1 \cdot c_2 + a_3 \cdot b_2 \cdot c_1 + a_3 \cdot b_3 \cdot c_2)$$
$$+(a_3 \cdot b_1 \cdot c_1 - a_3 \cdot b_2 \cdot c_2 - a_3 \cdot b_3 \cdot c_3).$$
$$(-a_2 \cdot b_2 \cdot c_3) + (a_3 \cdot b_2 \cdot c_2)$$
$$\neq (a_3 \cdot b_3 \cdot c_2) + (-a_3 \cdot b_2 \cdot c_2).$$

$((a_1, a_2, a_3) \odot (b_1, b_2, b_3)) \odot (c_1, c_2, c_3)$

$- (a_1, a_2, a_3) \odot ((b_1, b_2, b_3) \odot (c_1, c_2, c_3))$

$= (-(a_3 \cdot b_2 \cdot c_2) - (a_3 \cdot b_2 \cdot c_3) + (a_2 \cdot b_3 \cdot c_2) + (a_3 \cdot b_3 \cdot c_2),$

$(-a_3 \cdot b_3 \cdot c_2) + (a_3 \cdot b_2 \cdot c_2) + (a_2 \cdot b_3 \cdot c_3) - (a_3 \cdot b_3 \cdot c_2),$

$(-a_2 \cdot b_2 \cdot c_3) + (a_3 \cdot b_2 \cdot c_2) - (a_3 \cdot b_3 \cdot c_2) + (a_3 \cdot b_2 \cdot c_2)).$

C Proof of the Non-Alternativity of the Product $\odot$

A non-associative algebra is alternative if: $X(XY) = (XX)Y$ and $(YX)X = Y(XX)$ for all X, Y in the algebra. Defini

$b_2, b_3)$, we have:

$X(XY)$**:**

$(a_1, a_2, a_3) \odot (a_1 \cdot b_1 - a_2 \cdot b_2 - a_3 \cdot b_3, a_1 \cdot b_2 + a_2 \cdot b_1 + a_3 \cdot b_2, a_1 \cdot b_3 + a_3 \cdot b_2 + a_3 \cdot b_1) =$

$a_1 \cdot (a_1 \cdot b_1 - a_2 \cdot b_2 - a_3 \cdot b_3)$

$- a_2 \cdot (a_1 \cdot b_2 + a_2 \cdot b_1 + a_3 \cdot b_2)$

$- a_3 \cdot (a_1 \cdot b_3 + a_3 \cdot b_2 + a_3 \cdot b_1), a_1 \cdot (a_1 \cdot b_2 + a_2 \cdot b_1 + a_3 \cdot b_2$

$+ a_2 \cdot (a_1 \cdot b_1 - a_2 \cdot b_2 - a_3 \cdot b_3)$

$+ a_3 \cdot (a_1 \cdot b_2 + a_2 \cdot b_1 + a_3 \cdot b_2), a_1 \cdot (a_1 \cdot b_3 + a_3 \cdot b_2 + a_3 \cdot b_1$

$+ a_3 \cdot (a_1 \cdot b_2 + a_2 \cdot b_1 + a_3 \cdot b_2)$

$+ a_3 \cdot (a_1 \cdot b_1 - a_2 \cdot b_2 - a_3 \cdot b_3)\ .$

$(XX)Y$**:**

$(a_1 \cdot a_1 - a_2 \cdot a_2 - a_3 \cdot a_3, a_1 \cdot a_2 + a_2 \cdot a_1 + a_3 \cdot a_2, a_1 \cdot a_3 + a_3 \cdot a_2 + a_3 \cdot a_1) \odot (b_1, b_2, b_3) =$

$((a_1 \cdot a_1 - a_2 \cdot a_2 - a_3 \cdot a_3) \cdot b_1$

$- (a_1 \cdot a_2 + a_2 \cdot a_1 + a_3 \cdot a_2) \cdot b_2$

$- (a_1 \cdot a_3 + a_3 \cdot a_2 + a_3 \cdot a_1) \cdot b_3,$

$(a_1 \cdot a_1 - a_2 \cdot a_2 - a_3 \cdot a_3) \cdot b_2$

$+ (a_1 \cdot a_2 + a_2 \cdot a_1 + a_3 \cdot a_2) \cdot b_1$

$+ (a_1 \cdot a_3 + a_3 \cdot a_2 + a_3 \cdot a_1) \cdot b_2,$

$(a_1 \cdot a_1 - a_2 \cdot a_2 - a_3 \cdot a_3) \cdot b_3$

$+ (a_1 \cdot a_3 + a_3 \cdot a_2 + a_3 \cdot a_1) \cdot b_2$

$+ (a_1 \cdot a_3 + a_3 \cdot a_2 + a_3 \cdot a_1) \cdot b_1.$

$X(XY) - (XX)Y$:

$X(XY) - (XX)Y =$

$(-a_3 \cdot a_3 \cdot b_2 + a_3 \cdot a_2 \cdot b_3, -a_2 \cdot a_3 \cdot b_3 + a_3 \cdot a_3 \cdot b_2 + a_3 \cdot a_3 \cdot b_2 - a_3 \cdot a_2 \cdot b_2, a_3 \cdot a_3 \cdot b_2 - a_3 \cdot a_2 \cdot b_2 + a2 \cdot a2 \cdot b_3 - a_3 \cdot a_2 \cdot b_2).$

$Y(XX)$:

$(b_1,b_2,b_3) \odot (a_1 \cdot a_1 - a_2 \cdot a_2 - a_3 \cdot a_3, a_1 \cdot a_2 + a_2 \cdot a_1 + a_3 \cdot a_2, a_1 \cdot a_3 + a_3 \cdot a_2 + a_3 \cdot a_1) =$

$$\begin{aligned}
&(b_1 \cdot (a_1 \cdot a_1 - a_2 \cdot a_2 - a_3 \cdot a_3)\\
&- b_2 \cdot (a_1 \cdot a_2 + a_2 \cdot a_1 + a_3 \cdot a_2)\\
&- b_3 \cdot (a_1 \cdot a_3 + a_3 \cdot a_2 + a_3 \cdot a_1),\\
&b_1 \cdot (a_1 \cdot a_2 + a_2 \cdot a_1 + a_3 \cdot a_2)\\
&+ b_2 \cdot (a_1 \cdot a_1 - a_2 \cdot a_2 - a_3 \cdot a_3)\\
&+ b_3 \cdot (a_1 \cdot a_2 + a_2 \cdot a_1 + a_3 \cdot a_2),\\
&b_1 \cdot (a_1 \cdot a_3 + a_3 \cdot a_2 + a_3 \cdot a_1)\\
&+ b_2 \cdot (a_1 \cdot a_2 + a_2 \cdot a_1 + a_3 \cdot a_2)\\
&+ b_3 \cdot (a_1 \cdot a_1 - a_2 \cdot a_2 - a_3 \cdot a_3)).
\end{aligned}$$

(Y X)X:

$(b_1 \cdot a_1 - b_2 \cdot a_2 - b_3 \cdot a_3, b_1 \cdot a_2 + b_2 \cdot a_1 + b_3 \cdot a_2, b_1 \cdot a_3 + b_3 \cdot a_2 + b_3 \cdot a_1) \odot (a_1, a_2, a_3) =$

$$\begin{aligned}
&((b_1 \cdot a_1 - b_2 \cdot a_2 - b_3 \cdot a_3) \cdot a_1\\
&- (b_1 \cdot a_2 + b_2 \cdot a_1 + b_3 \cdot a_2) \cdot a_2\\
&- (b_1 \cdot a_3 + b_3 \cdot a_2 + b_3 \cdot a_1) \cdot a_3,\\
&(b_1 \cdot a_1 - b_2 \cdot a_2 - b_3 \cdot a_3) \cdot a_2\\
&+ (b_1 \cdot a_2 + b_2 \cdot a_1 + b_3 \cdot a_2) \cdot a_1\\
&+ (b_1 \cdot a_3 + b_3 \cdot a_2 + b_3 \cdot a_1) \cdot a_2,\\
&(b_1 \cdot a_1 - b_2 \cdot a_2 - b_3 \cdot a_3) \cdot a_3\\
&+ (b_1 \cdot a_3 + b_3 \cdot a_2 + b_3 \cdot a_1) \cdot a_2\\
&+ (b_1 \cdot a_3 + b_3 \cdot a_2 + b_3 \cdot a_1) \cdot a_1).
\end{aligned}$$

(Y X)X − Y (XX):

$$(Y X)X - Y (XX) =$$

$(-b_2 \cdot a_3 \cdot a_2 + b_3 \cdot a_2 \cdot a_2, -b_2 \cdot a_3 \cdot a_3 + b_3 \cdot a_3 \cdot a_2 + b_3 \cdot a_3 \cdot a_2 - b_3 \cdot a_2 \cdot a_2, b_3 \cdot a_3 \cdot a_2 - b_3 \cdot a_2 \cdot a_2 + b_2 \cdot a_2 \cdot a_3 - b_3 \cdot a_2 \cdot a_2)$.

D Proof of the Bilinearity of the Lie Bracket defined from $\odot$

Bilinearity holds if, for all $\lambda, \gamma \in F$:

$[\lambda \cdot (a_1, a_2, a_3) + \gamma \cdot (b_1, b_2, b_3), (c_1, c_2, c_3)] = \lambda \cdot [(a_1, a_2, a_3), (c_1, c_2, c_3)] + \gamma \cdot [(b_1, b_2, b_3), (c_1, c_2, c_3)]$.

Left-Hand Side:

$[\lambda \cdot (a_1, a_2, a_3) + \gamma \cdot (b_1, b_2, b_3), (c_1, c_2, c_3)] =$

$\lambda \cdot (a_1, a_2, a_3) + \gamma \cdot (b_1, b_2, b_3) \odot (c_1, c_2, c_3) - (c_1, c_2, c_3) \odot \lambda \cdot (a_1, a_2, a_3) + \gamma \cdot (b_1, b_2, b_3)$.

First Term:

$$\lambda \cdot (a_1, a_2, a_3) + \gamma \cdot (b_1, b_2, b_3) \odot (c_1, c_2, c_3)$$

$$= (\lambda \cdot a_1 + \gamma \cdot b_1, \lambda \cdot a_2 + \gamma \cdot b_2, \lambda \cdot a_3 + \gamma \cdot b_3) \odot (c_1, c_2, c_3)$$

$$= \ (\lambda \cdot a_1 + \gamma \cdot b_1) \cdot c_1 - (\lambda \cdot a_2 + \gamma \cdot b_2) \cdot c_2 - (\lambda \cdot a_3 + \gamma \cdot b_3) \cdot c_3,$$

$$(\lambda \cdot a_1 + \gamma \cdot b_1) \cdot c_2 + (\lambda \cdot a_2 + \gamma \cdot b_2) \cdot c_1 + (\lambda \cdot a_3 + \gamma \cdot b_3) \cdot c_2,$$

$$(\lambda \cdot a_1 + \gamma \cdot b_1) \cdot c_3 + (\lambda \cdot a_3 + \gamma \cdot b_3) \cdot c_2 + (\lambda \cdot a_3 + \gamma \cdot b_3) \cdot c_1$$

$$= \ \lambda \cdot a_1 \cdot c_1 + \gamma \cdot b_1 \cdot c_1 - \lambda \cdot a_2 \cdot c_2 - \gamma \cdot b_2 \cdot c_2 - \lambda \cdot a_3 \cdot c_3 - \gamma \cdot b_3 \cdot c_3,$$

$$\lambda \cdot a_1 \cdot c_2 + \gamma \cdot b_1 \cdot c_2 + \lambda \cdot a_2 \cdot c_1 + \gamma \cdot b_2 \cdot c_1 + \lambda \cdot a_3 \cdot c_2 + \gamma \cdot b_3 \cdot c_2,$$

$$\lambda \cdot a_1 \cdot c_3 + \gamma \cdot b_1 \cdot c_3 + \lambda \cdot a_3 \cdot c_2 + \gamma \cdot b_3 \cdot c_2 + \lambda \cdot a_3 \cdot c_1 + \gamma \cdot b_3 \cdot c_1 \ .$$

Second Term:

$$(c_1, c_2, c_3) \odot \lambda \cdot (a_1, a_2, a_3) + \gamma \cdot (b_1, b_2, b_3)$$

$$= (c_1, c_2, c_3) \odot (\lambda \cdot a_1 + \gamma \cdot b_1, \lambda \cdot a_2, \gamma \cdot b_2, \lambda \cdot a_3, \gamma \cdot b_3)$$

$$= \ c_1 \cdot (\lambda \cdot a_1 + \gamma \cdot b_1) - c_2 \cdot (\lambda \cdot a_2 + \gamma \cdot b_2) - c_3 \cdot (\lambda \cdot a_3 + \gamma \cdot b_3),$$

$$c_1 \cdot (\lambda \cdot a_2 + \gamma \cdot b_2) + c_2 \cdot (\lambda \cdot a_1 + \gamma \cdot b_1) + c_3 \cdot (\lambda \cdot a_2 + \gamma \cdot b_2),$$

$$c_1 \cdot (\lambda \cdot a_3 + \gamma \cdot b_3) + c_3 \cdot (\lambda \cdot a_2 + \gamma \cdot b_2) + c_3 \cdot (\lambda \cdot a_1 + \gamma \cdot b_1)$$

$$= (c_1 \cdot \lambda \cdot a_1 + c_1 \cdot \gamma \cdot b_1 - c_2 \cdot \lambda \cdot a_2 - c_2 \cdot \gamma \cdot b_2 - c_3 \cdot \lambda \cdot a_3 - c_3 \cdot \gamma \cdot b_3,$$

$$c_1 \cdot \lambda \cdot a_2 + c_1 \cdot \gamma \cdot b_2 + c_2 \cdot \lambda \cdot a_1 + c_2 \cdot \gamma \cdot b_1 + c_3 \cdot \lambda \cdot a_2 + c_3 \cdot \gamma \cdot b_2,$$

$$c_1 \cdot \lambda \cdot a_3 + c_1 \cdot \gamma \cdot b_3 + c_3 \cdot \lambda \cdot a_2 + c_3 \cdot \gamma \cdot b_2 + c_3 \cdot \lambda \cdot a_1 + c_3 \cdot \gamma \cdot b_1).$$

First Term - Second Term:

$$(\lambda \cdot a_1 \cdot c_1 + \gamma \cdot b_1 \cdot c_1 - \lambda \cdot a_2 \cdot c_2 - \gamma \cdot b_2 \cdot c_2 - \lambda \cdot a_3 \cdot c_3 - \gamma \cdot b_3 \cdot c_3,$$

$$\lambda \cdot a_1 \cdot c_2 + \gamma \cdot b_1 \cdot c_2 + \lambda \cdot a_2 \cdot c_1 + \gamma \cdot b_2 \cdot c_1 + \lambda \cdot a_3 \cdot c_2 + \gamma \cdot b_3 \cdot c_2,$$

$$\lambda \cdot a_1 \cdot c_3 + \gamma \cdot b_1 \cdot c_3 + \lambda \cdot a_3 \cdot c_2 + \gamma \cdot b_3 \cdot c_2 + \lambda \cdot a_3 \cdot c_1 + \gamma \cdot b_3 \cdot c_1)$$

$$-(c_1 \cdot \lambda \cdot a_1 + c_1 \cdot \gamma \cdot b_1 - c_2 \cdot \lambda \cdot a_2 - c_2 \cdot \gamma \cdot b_2 - c_3 \cdot \lambda \cdot a_3 - c_3 \cdot \gamma \cdot b_3,$$

$$c_1 \cdot \lambda \cdot a_2 + c_1 \cdot \gamma \cdot b_2 + c_2 \cdot \lambda \cdot a_1 + c_2 \cdot \gamma \cdot b_1 + c_3 \cdot \lambda \cdot a_2 + c_3 \cdot \gamma \cdot b_2,$$

$$c_1 \cdot \lambda \cdot a_3 + c_1 \cdot \gamma \cdot b_3 + c_3 \cdot \lambda \cdot a_2 + c_3 \cdot \gamma \cdot b_2 + c_3 \cdot \lambda \cdot a_1 + c_3 \cdot \gamma \cdot b_1)$$

$$= (0, \lambda \cdot a_3 \cdot c_2 + \gamma \cdot b_3 \cdot c_2 - c_3 \cdot \lambda \cdot a_2 - c_3 \cdot \gamma \cdot b_2,$$

$$\lambda \cdot a_3 \cdot c_2 + \gamma \cdot b_3 \cdot c_2 - c_3 \cdot \lambda \cdot a_2 - c_3 \cdot \gamma \cdot b_2).$$

Right-Hand Side:

$$\lambda \cdot [(a_1, a_2, a_3), (c_1, c_2, c_3)] + \gamma \cdot [(b_1, b_2, b_3), (c_1, c_2, c_3)].$$

First Term:

$\lambda \cdot [(a_1, a_2, a_3), (c_1, c_2, c_3)] = \lambda \cdot (a_1 \cdot c_1 - a_2 \cdot c_2 - a_3 \cdot c_3,$

$a_1 \cdot c_2 + a_2 \cdot c_1 + a_3 \cdot c_2,$

$a_1 \cdot c_3 + a_3 \cdot c_2 + a_3 \cdot c_1)$

$- \lambda \cdot (c_1 \cdot a_1 - c_2 \cdot a_2 - c_3 \cdot a_3,$

$c_1 \cdot a_2 + c_2 \cdot a_1 + c_3 \cdot a_2,$

$c_1 \cdot a_3 + c_3 \cdot a_2 + c_3 \cdot a_1)$

$= (0, \lambda \cdot a_3 \cdot c_2 - \lambda \cdot c_3 \cdot a_2,$

$\lambda \cdot a_3 \cdot c_2 - \lambda \cdot c_3 \cdot a_2)$

$= \lambda \cdot (a_3 \cdot c_2 - c_3 \cdot a_2) \cdot (0, 1, 1).$

Second Term: $\gamma \cdot [(b_1, b_2, b_3), (c_1, c_2, c_3)] = \gamma \cdot (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot (0, 1, 1).$

First Term + Second Term:

$\lambda \cdot (a_3 \cdot c_2 - c_3 \cdot a_2) \cdot (0, 1, 1) + \gamma \cdot (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot (0, 1, 1)$

$= (0, \lambda \cdot (a_3 \cdot c_2 - c_3 \cdot a_2), \lambda \cdot (a_3 \cdot c_2 - c_3 \cdot a_2))$

$+ (0, \gamma \cdot (b_3 \cdot c_2 - c_3 \cdot b_2), \gamma \cdot (b_3 \cdot c_2 - c_3 \cdot b_2))$

$= (0, \lambda \cdot (a_3 \cdot c_2 - c_3 \cdot a_2) + \gamma \cdot (b_3 \cdot c_2 - c_3 \cdot b_2),$

$\lambda \cdot (a_3 \cdot c_2 - c_3 \cdot a_2) + \gamma \cdot (b_3 \cdot c_2 - c_3 \cdot b_2))$

$= (\lambda \cdot (a_3 \cdot c_2 - c_3 \cdot a_2) + \gamma \cdot (b_3 \cdot c_2 - c_3 \cdot b_2)) \cdot (0, 1, 1).$

Therefore:

$(0, \lambda \cdot a_3 \cdot c_2 + \gamma \cdot b_3 \cdot c_2 - c_3 \cdot \lambda \cdot a_2 - c_3 \cdot \gamma \cdot b_2,$

$\lambda \cdot a_3 \cdot c_2 + \gamma \cdot b_3 \cdot c_2 - c_3 \cdot \lambda \cdot a_2 - c_3 \cdot \gamma \cdot b_2)$

$= (0, \lambda \cdot a_3 \cdot c_2 - \lambda \cdot c_3 \cdot a_2 + \gamma \cdot b_3 \cdot c_2 - \gamma \cdot c_3 \cdot b_2,$

$\lambda \cdot a_3 \cdot c_2 - \lambda \cdot c_3 \cdot a_2 + \gamma \cdot b_3 \cdot c_2 - \gamma \cdot c_3 \cdot b_2).$

E Proof of the Jacobi Identity

The Jacobi identity is satisfied when:

$$[(a_1, a_2, a_3), [(b_1, b_2, b_3), (c_1, c_2, c_3)]] + [(b_1, b_2, b_3), [(c_1, c_2, c_3), (a_1, a_2, a_3)]] + [(c_1, c_2, c_3), [(a_1, a_2, a_3), (b_1, b_2, b_3)]] = (0, 0, 0).$$

Proof:

$[(b_1, b_2, b_3), (c_1, c_2, c_3)] = (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot (0, 1, 1) = (0, b_3 \cdot c_2 - c_3 \cdot b_2, b_3 \cdot c_2 - c_3 \cdot b_2).$

$[(a_1, a_2, a_3), [(b_1, b_2, b_3), (c_1, c_2, c_3)]] = (a_3 \cdot (b_3 \cdot c_2 - c_3 \cdot b_2) - (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot a_2) \cdot (0,1,1)$. $[(a_1, a_2, a_3), [(b_1, b_2, b_3), (c_1$

$= (a_1, a_2, a_3) \odot (0, b_3 \cdot c_2 - c_3 \cdot b_2, b_3 \cdot c_2 - c_3 \cdot b_2)$

$- (0, b_3 \cdot c_2 - c_3 \cdot b_2, b_3 \cdot c_2 - c_3 \cdot b_2) \odot (a_1, a_2, a_3)$.

$(a_1, a_2, a_3) \odot (0, b_3 \cdot c_2 - c_3 \cdot b_2, b_3 \cdot c_2 - c_3 \cdot b_2)$:

$$(a_1, a_2, a_3) \odot (0, b_3 \cdot c_2 - c_3 \cdot b_2, b_3 \cdot c_2 - c_3 \cdot b_2)$$
$$= (-a_2 \cdot (b_3 \cdot c_2 - c_3 \cdot b_2) - a_3 \cdot (b_3 \cdot c_2 - c_3 \cdot b_2),$$
$$a_1 \cdot (b_3 \cdot c_2 - c_3 \cdot b_2) + a_3 \cdot (b_3 \cdot c_2 - c_3 \cdot b_2),$$
$$a_1 \cdot (b_3 \cdot c_2 - c_3 \cdot b_2) + a_3 \cdot (b_3 \cdot c_2 - c_3 \cdot b_2)).$$

$(0, b_3 \cdot c_2 - c_3 \cdot b_2, b_3 \cdot c_2 - c_3 \cdot b_2) \odot (a_1, a_2, a_3)$:

$$(0, b_3 \cdot c_2 - c_3 \cdot b_2, b_3 \cdot c_2 - c_3 \cdot b_2) \odot (a_1, a_2, a_3)$$
$$= (-(b_3 \cdot c_2 - c_3 \cdot b_2) \cdot a_2 - (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot a_3,$$
$$(b_3 \cdot c_2 - c_3 \cdot b_2) \cdot a_1 + (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot a_2,$$
$$(b_3 \cdot c_2 - c_3 \cdot b_2) \cdot a_2 + (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot a_1).$$

$$(-a_2 \cdot b_3 \cdot c_2 + a_2 \cdot c_3 \cdot b_2 - a_3 \cdot b_3 \cdot c_2 + a_3 \cdot c_3 \cdot b_2, a_1 \cdot b_3 \cdot c_2 - a_1 \cdot c_3 \cdot b_2 + a_3 \cdot b_3 \cdot c_2 - a_3 \cdot c_3 \cdot b_2, a_1 \cdot b_3 \cdot c_2 - a_1 \cdot c_3 \cdot b_2 + a$$
$$- (-b_3 \cdot c_2 \cdot a_2 + c_3 \cdot b_2 \cdot a_2 - b_3 \cdot c_2 \cdot a_3 + c_3 \cdot b_2 \cdot a_3, b_3 \cdot c_2 \cdot a_1 - c_3 \cdot b_2 \cdot a_1 + b_3 \cdot c_2 \cdot a_2 - c_3 \cdot b_2 \cdot a_2,$$
$$b_3 \cdot c_2 \cdot a_2 - c_3 \cdot b_2 \cdot a_2 + b_3 \cdot c_2 \cdot a_1 - c_3 \cdot b_2 \cdot a_1)$$
$$= (0, a_3 \cdot (b_3 \cdot c_2 - c_3 \cdot b_2) - (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot a_2, a_3 \cdot (b_3 \cdot c_2 - c_3 \cdot b_2) - (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot a_2)$$
$$= (a_3 \cdot (b_3 \cdot c_2 - c_3 \cdot b_2) - (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot a_2) \cdot (0, 1, 1).$$

Continuation of the Proof:

$$[(b_1, b_2, b_3), [(c_1, c_2, c_3), (a_1, a_2, a_3)]]$$
$$= (b_3 \cdot (c_3 \cdot a_2 - a_3 \cdot c_2) - (c_3 \cdot a_2 - a_3 \cdot c_2) \cdot b_2) \cdot (0, 1, 1),$$
$$[(c_1, c_2, c_3), [(a_1, a_2, a_3), (b_1, b_2, b_3)]]$$
$$= (c_3 \cdot (a_3 \cdot b_2 - b_3 \cdot a_2) - (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot c_2) \cdot (0, 1, 1).$$
$$(a_3 \cdot (b_3 \cdot c_2 - c_3 \cdot b_2) - (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot a_2) \cdot (0, 1, 1)$$
$$+ (b_3 \cdot (c_3 \cdot a_2 - a_3 \cdot c_2) - (c_3 \cdot a_2 - a_3 \cdot c_2) \cdot b_2) \cdot (0, 1, 1)$$
$$+ (c_3 \cdot (a_3 \cdot b_2 - b_3 \cdot a_2) - (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot c_2) \cdot (0, 1, 1)$$
$$= (a_3 \cdot (b_3 \cdot c_2 - c_3 \cdot b_2) - (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot a_2) \cdot (0, 1, 1)$$
$$+ (b_3 \cdot (c_3 \cdot a_2 - a_3 \cdot c_2) - (c_3 \cdot a_2 - a_3 \cdot c_2) \cdot b_2) \cdot (0, 1, 1)$$
$$+ (c_3 \cdot (a_3 \cdot b_2 - b_3 \cdot a_2) - (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot c_2) \cdot (0, 1, 1)$$

$= ((a_3 \cdot b_3 \cdot c_2 - a_3 \cdot c_3 \cdot b_2) - (b_3 \cdot c_2 \cdot a_2 - c_3 \cdot b_2 \cdot a_2)) \cdot (0, 1, 1)$

$+ ((b_3 \cdot c_3 \cdot a_2 - b_3 \cdot a_3 \cdot c_2) - (c_3 \cdot a_2 \cdot b_2 - a_3 \cdot c_2 \cdot b_2)) \cdot (0, 1, 1)$

$+ ((c_3 \cdot a_3 \cdot b_2 - c_3 \cdot b_3 \cdot a_2) - (a_3 \cdot b_2 \cdot c_2 - b_3 \cdot a_2 \cdot c_2)) \cdot (0, 1, 1)$

$= (a_3 \cdot b_3 \cdot c_2 - a_3 \cdot c_3 \cdot b_2 - b_3 \cdot c_2 \cdot a_2 + c_3 \cdot b_2 \cdot a_2$

$+ b_3 \cdot c_3 \cdot a_2 - b_3 \cdot a_3 \cdot c_2 - c_3 \cdot a_2 \cdot b_2 + a_3 \cdot c_2 \cdot b_2$

$+ c_3 \cdot a_3 \cdot b_2 - c_3 \cdot b_3 \cdot a_2 - a_3 \cdot b_2 \cdot c_2 + b_3 \cdot a_2 \cdot c_2) \cdot (0, 1, 1)$

$= (0, 0, 0).$

F Proof of the Malcev Identity

The Malcev identity holds if: $\forall a, b, c \in M : [[a, b], [a, c]] = [[[a, b], c], a] + [[[b, c], a], a] + [[[c, a], a], b]$.

Defining these a, b, c as $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, and $c = (c_1, c_2, c_3)$, we have:

$$[a, b] = (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (0, 1, 1).$$

$$[a, c] = (a_3 \cdot c_2 - c_3 \cdot a_2) \cdot (0, 1, 1).$$

$[[a, b], [a, c]]$

$= (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (0, 1, 1) \odot (a_3 \cdot c_2 - c_3 \cdot a_2) \cdot (0, 1, 1)$

$— (a_3 \cdot c_2 - c_3 \cdot a_2) \cdot (0, 1, 1) \odot (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (0, 1, 1)$

$= (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (a_3 \cdot c_2 - c_3 \cdot a_2) \cdot (0, 1, 1) \odot (0, 1, 1)$

$— (a_3 \cdot c_2 - c_3 \cdot a_2) \cdot (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (0, 1, 1) \odot (0, 1, 1)$

$= (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (a_3 \cdot c_2 - c_3 \cdot a_2) \cdot (-1 - 1, 1, 1)$

$— (a_3 \cdot c_2 - c_3 \cdot a_2) \cdot (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (-1 - 1, 1, 1)$

$= ((a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (a_3 \cdot c_2 - c_3 \cdot a_2)$

$— (a_3 \cdot c_2 - c_3 \cdot a_2) \cdot (a_3 \cdot b_2 - b_3 \cdot a_2)) \cdot (-1 - 1, 1, 1)$

$= (0, 0, 0).$

$[[a, b], c] = [(a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (c_2 - c_3) \cdot (0, 1, 1)].$

$[[[a, b], c], a]$

$= [(a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (c_2 - c_3) \cdot (0, 1, 1) \odot (a_1, a_2, a_3)$

$— (a_1, a_2, a_3) \odot (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (c_2 - c_3) \cdot (0, 1, 1)]$

$= (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (c_2 - c_3) \cdot$

$[((-a_2 - a_3, a_1 + a_2, a_2 + a_1) - (-a_2 - a_3, a_1 + a_3, a_1 + a_3))]$

$= (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (c_2 - c_3) \cdot (0, a_2 - a_3, a_2 - a_3)$

$= (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (c_2 - c_3) \cdot (a_2 - a_3) \cdot (0, 1, 1).$

$[[b, c], a] = (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot (a_2 - a_3) \cdot (0, 1, 1).$

$[[[b, c], a], a]$

$= (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot (a_2 - a_3) \cdot (0, 1, 1) \odot (a_1, a_2, a_3)$

$— (a_1, a_2, a_3) \odot (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot (a_2 - a_3) \cdot (0, 1, 1)$

$= (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot (a_2 - a_3)$

$\times [((-a_2 - a_3, a_1 + a_2, a_2 + a_1) - (-a_2 - a_3, a_1 + a_3, a_1 + a_3))]$

$= (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot (a_2 - a_3) \cdot (0, a_2 - a_3, a_2 - a_3)$

$= (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot (a_2 - a_3) \cdot (a_2 - a_3) \cdot (0, 1, 1).$

$[[c, a], a] = (c_3 \cdot a_2 - a_3 \cdot c_2) \cdot (a_2 - a_3) \cdot (0, 1, 1).$

$[[[c, a], a], b]$

$= (c_3 \cdot a_2 - a_3 \cdot c_2) \cdot (a_2 - a_3) \cdot (0, 1, 1) \odot (b_1, b_2, b_3)$

$— (b_1, b_2, b_3) \odot (c_3 \cdot a_2 - a_3 \cdot c_2) \cdot (a_2 - a_3) \cdot (0, 1, 1)$

$= (c_3 \cdot a_2 - a_3 \cdot c_2) \cdot (a_2 - a_3)$

$\times [((-b_2 - b_3, b_1 + b_2, b_2 + b_1) - (-b_2 - b_3, b_1 + b_3, b_1 + b_3))]$

$= (c_3 \cdot a_2 - a_3 \cdot c_2) \cdot (a_2 - a_3) \cdot (b_2 - b_3) \cdot (0, 1, 1).$

$[[[a, b], c], a] + [[[b, c], a], a] + [[[c, a], a], b]$

$= (a_3 \cdot b_2 - b_3 \cdot a_2) \cdot (c_2 - c_3) \cdot (a_2 - a_3) \cdot (0, 1, 1)$

$+ (b_3 \cdot c_2 - c_3 \cdot b_2) \cdot (a_2 - a_3) \cdot (a_2 - a_3) \cdot (0, 1, 1)$

$+ (c_3 \cdot a_2 - a_3 \cdot c_2) \cdot (a_2 - a_3) \cdot (b_2 - b_3) \cdot (0, 1, 1)$

$= (a_3 \cdot b_2 \cdot c_2 - a_3 \cdot b_2 \cdot c_3 - b_3 \cdot a_2 \cdot c_2 + b_3 \cdot a_2 \cdot c_3$

$+ b_3 \cdot c_2 \cdot a_2 - b_3 \cdot c_2 \cdot a_3 - c_3 \cdot b_2 \cdot a_2 + c_3 \cdot b_2 \cdot a_3$

$+ c_3 \cdot a_2 \cdot b_2 - c_3 \cdot a_2 \cdot b_3 - a_3 \cdot c_2 \cdot b_2 + a_3 \cdot c_2 \cdot b_3) \cdot (a_2 - a_3) \cdot (0, 1, 1)$

$= (0, 0, 0).$

$[[a, b], [a, c]] = [[[a, b], c], a] + [[[b, c], a], a] + [[[c, a], a], b] = (0, 0, 0).$