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Dimensional Analysis in General Theory of Relativity

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Abstract
Use of natural unit is utmost important for research in high energy physics, particularly in the fields of 'Elementary Particle Physics', 'Astro-particle Physics' and `Physical Cosmology', because all the parameters turn out to be dimensionless. However, it is suggestive to work in a system of units, in which all the kinematical parameters may be expressed in terms of different powers of the Planck's mass or equivalently the energy. Here we explicitly exhibit the use of the natural units, some associated problems and also the proposed system of units in connection with their application in 'General Theory of Relativity'.

1. Introduction

Out of the four fundamental SI units, three are kinematical, such as the metre (m), second (t) and kilogram (*kg*). In the natural unit, apart from the speed of light in vacuum (c = 1), the reduced Planck's constant ($\hbar$ = 1) and the Newtonian gravitational constant ($8\pi G$=1) are fixed and set to unity. In the process, the length, the time and the mass are fixed once and for ever and all the parameters are dimensionless. This creates some problems in the perturbative analysis of a quantum system. On the contrary, if G or equivalently mass/energy is not fixed to unity, then all the parameters may be expressed in terms of different powers of a single unit, viz., the mass or equivalently the energy. This not only simplifies the determination of the degrees of the units of additional parameters if introduced in the equations, but also helps to compare the magnitude of different parameters of the theory with observational data, without any problem. Here, we explicitly compute to exhibit the use of natural unit (which makes all the parameters dimensionless) and also the system of units in which all the parameters can be expressed in terms of different powers of the mass/energy, i.e., the system in which Newton's gravitational constant G is not fixed to unity.

Before we proceed, let us first recapitulate the constants associated with Planck's scale. The three fundamental constants are the Planck's mass M_P , the Planck's length l_P and the Planck's time t_P. At the Planck's scale, `General Theory of Relativity' (GTR) and consequently, the `Standard Model of Cosmology' (FLRW model) collapse and there is the need for a `Quantum Theory of Gravity', which is not at hand presently, despite tremendous effort over several decades. This implies that we have no present knowledge of physics at the Planck's scale and beyond. 'Astro-particle Physics' and 'Physical Cosmology' entail the implication of these three fundamental constants of nature, which are: Planck's mass ($M_P \approx 10^{-19}\ GeV$) is the minimum possible mass and consequently the minimum size $\left(r = \frac{2GM}{c^2}\right)$ of a black hole (it may also be interpreted as, no fundamental particle can have mass greater than the Planck's mass); no physical experiment can ever probe beyond Planck's length ($l_P \approx 10^{-33}\ cm$). (it is the smallest possible measure of space) and finally, it is debarred to ask what happened before Planck's time ($t_P \approx 10^{-43}\ s$), while the smallest time-interval measured so far is $10^{-21}\ s$, respectively.

The manuscript is organized as follows. In the following section, we make dimensional analysis of the physical parameters involved in GTR and also for some modified theories of gravity. Next, in section 3 we shall compute numerical values of the three fundamental Planck's scale constants, in view of the known numerical values of the reduced Planck's constant $\left(\hbar = \frac{h}{2\pi}\right)$, the Newtonian gravitational constant (G) and the velocity of light in vacuum c. Additionally, we compute the Planck's temperature (T_P) using the numerical value of the Boltzmann constant (k_B). In section 4, we discuss the use of natural unit in GTR along with some associated problems. Section 5 is devoted to work in a system of units in which all the parameters may be expressed in terms of the Planck's mass. A brief conclusion appears in section 6.

2. Dimensions of the Parameters Involved in GTR

The Einstein's equation for the `General Theory of Relativity' (GTR) in the presence of a cosmological constant Λ is given by

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu} \tag{1}$$

where, $G_{\mu\nu}$, $R_{\mu\nu}$, R, $g_{\mu\nu}$, $T_{\mu\nu}$ are the Einstein tensor, the Ricci tensor, the Ricci scalar, the metric tensor and the stress-energy tensor respectively, G is the Newton's gravitational constant while c is the velocity of light. In 4-dimensional space-time, μ, ν run from 0-3. It may be mentioned that the constant $\kappa = \frac{8\pi G}{c^4}$ had been so chosen that under non-relativistic limit, Newton's law of gravitation, Poisson equation ($\nabla^2 \phi = 4\pi G\rho$, ρ being the mass-density) to be specific, rejuvenates. For dimensional analysis let us start from the space-time metric,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

Since the dimension of the left-hand side is L^2, so clearly, $g_{\mu\nu}$ is dimensionless. As a result. contraction of a tensor by the metric tensor does not change its dimension. Now the connection co-efficient, viz., the Levi-Civita connection given by,

$$\Gamma^\mu_{\beta\gamma} = \frac{1}{2} g^{\mu\alpha}\left(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}\right) \tag{3}$$

is essentially formed out of the coordinate derivative of the metric tensor and hence has dimension of L^{-1}. Further the Riemann tensor is again made up of the coordinate derivative of the connection tensor and is given by,

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\alpha_{\mu\gamma}\Gamma^\mu_{\beta\delta} - \Gamma^\alpha_{\mu\delta}\Gamma^\mu_{\beta\gamma} \tag{4}$$

Therefore, the Riemann tensor has the dimension L^{-2}. Under contraction one finds the Ricci tensor, which under further contraction gives the Ricci scalar and both have the dimension of L^{-2}. For dimensional matching, the dimension of $[\Lambda] = L^{-2}$. Thus we obtain the dimension of the left-hand side of Einstein equation (1), viz. $[G_{\mu\nu}] = L^{-2}\}$. Let us now look for the right-hand side. The dimension of the Newtonian gravitational constant is $[G] = L^3 M^{-1} T^{-2}$. The energy-momentum tensor is essentially the energy per unit volume which is the force per unit area or the pressure. Hence its dimension is $[T_{\mu\nu}] = ML^{-1} T^{-2}$. As a result, the dimension of the right-hand side of the Einstein's equation (1) is $\left[\frac{8\pi G}{c^4} T_{\mu\nu}\right] = L^3 M^{-1} T^{-2} \times L^{-4} T^4 \times ML^{-1} T^{-2} = L^{-2}$. For a cross-check, note that under contraction, the Einstein's equation (1) yields

$$-R + 4\Lambda = \frac{8\pi G}{c^4}\ \mathrm{T}, \text{ or equivalently, } \mathrm{T} = \frac{c^4}{8\pi G}(-R + 4\Lambda) \tag{5}$$

where, T is the trace of the energy-momentum tensor $T_{\mu\nu}$. Therefore, the dimension of the $[\mathrm{T}] = L^{-2} \times \left[\frac{c^4}{G}\right] = ML^{-1}T^{-2}$ and so is the dimension of the energy-momentum tensor too.

Now, in order to modify GTR (if required), one has to start from the following Einstein-Hilbert action,

$$A = \int \left[\frac{c^4}{16\pi G}\,(R - 2\Lambda)\right]\sqrt{-g}\, d^4x + S_m \tag{6}$$

where, $S_m = \int \mathcal{L}_m \sqrt{-g}\, d^4 x$ stands for the matter action, while $\mathcal{L}_m$ is the matter Lagrangian density. After integrating the spatial part, the action in its simplest form is given by $A = \int L\, dt$, where the point Lagrangian, $L = T - V$ has the dimension of energy. So, the action has the dimension of $[A] = M L^2 T^{-1}$. Here, one can check that the dimension of $\left[\frac{c^4}{G} R\right] = ML^{-1} T^{-2}$, while the integral is over the four-volume, whose dimension is $[d^4 x] = L^3 T$. Hence the dimension of the action becomes $[A] = ML^2 T^{-1}$.

If we now want to form a modified theory of gravity by adding additional curvature invariant terms such as R^2, R^{-1} etc, the action would then take the form [1],

$$A = \int \left[\frac{c^4}{16\pi G}\,(R - 2\Lambda) + \beta\, R^2 + \gamma\, R^{-1} + \mathcal{L}_m\right]\sqrt{-g}\, d^4x \tag{7}$$

Now in order to find the dimensions of the constant parameters β, and γ , we note that βR^2 and γR^{-1} must have the dimension of $\left[\frac{c^4}{G} R\right] = L^{-1} MT^{-2}$. Thus, dimension of $[\beta] = L^3 M T^{-2}$, and that of $[\gamma] = L^{-3} MT^{-2}$. Finally for a scalar field, the matter Lagrangian density is,

$$\mathcal{L}_m = -\frac{1}{2}\, g_{\mu\nu}\phi^{,\mu}\,\phi^{,\nu} - V(\phi) \tag{8}$$

following (-, +, +, +) or equivalently +2 sign convention. Since $g_{\mu\nu}$ is dimensionless, so $\mathcal{L}_m$ also has the dimension of $\left[\frac{c^4}{G}R\right]$ = $[\mathcal{L}_m] = L^{-1} MT^{-2}$. To find the dimension of the scalar field (ϕ), we express $\mathcal{L}_m$ in Minkowskian metric ($\eta_{\mu\nu}$) which eases the situation,

$$[\mathcal{L}_m] = \left[\frac{1}{2}\left(\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{\partial\phi}{\partial x^i}\right)^2\right] - V(\phi) = L^{-1}MT^{-2} \tag{9}$$

$$[\eta] = [\phi^{-1}]$$

where, i runs from 1-3, implying spatial components. Now, $[c^2\ dt^2] = [(dx^i)^2] = L^2$, hence the dimension of ϕ is $[\phi] = L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1}$. In fact, using the definition of the square of the scalar field, which is the energy per unit length, one can also cross-check to find $[\phi]$= $\left[\sqrt{\frac{\text{Energy}}{\text{Length}}}\right] = \sqrt{ML^2T^{-2}L^{-1}} = L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}$. It is now quite trivial to find that the dimension of the constant parameter V_0 of a quadratic potential $V = V_0\,\phi^2$, which is $[V_0] = L^{-1}\ MT^{-2} \times M^{-1}\ L^{-1}\ T^2 = L^{-2}$. Likewise, the dimension of λ appearing in the quartic potential $\lambda\,\phi^4$ is $[\lambda] = L^{-3}\ M^{-1}\ T^2$, the dimension of V_0 appearing in the exponential potential $V_0\ e^{\pm\eta\phi}$ is $[V_0] = L^{-1}\ M\,T^{-2}$, and that of $[\eta]$ = $[\phi^{-1}]$, since exponent has to be dimensionless. Finally for a potential in the form $V(\phi) = V_4\ \phi^n$, the dimension of $[V_4]$= $M^{1-\frac{n}{2}}L^{-\left(1+\frac{n}{2}\right)}$ T^{n-2}, where n is any arbitrary real number. In this manner, one can find the dimensions of the constant param eters associated with different types of potentials. It may be mentioned that for a Brans-Dicke type scalar field,

$$\mathcal{L}_m = -\frac{\omega(\phi)}{2}\, g_{\mu\nu}\phi^{,\mu}\phi^{,\nu} - V(\phi) \tag{10}$$

the Brans-Dicke parameter $\omega(\phi)$ has to be dimensionless, regardless it be a constant or not. Table-1 below presents a glimpse of the dimensions discussed in this section.

Lagrangian	Lagrangian density	Action	Ricci scalar, Ricci tensor, Cosmological constant	Energy-Momentum Tensor	Scalar field	Potential
$[L]$	$[\mathcal{L}_m]$	$[A]$	$[R], [R_{\mu\nu}], [\Lambda]$	$[T_{\mu\nu}]$	$[\phi]$	$[V(\phi)]$
ML^2T^{-2}	$ML^{-1}T^{-2}$	ML^2T^{-1}	L^{-2}	$ML^{-1}T^{-2}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	$ML^{-1}T^{-2}$

Table 1: List of Dimensions of Some Standard Parameters Used in GTR

3. Planck's Scale

A system of units defined in terms of the four fundamental constants of nature such as, the velocity of light in vacuum (c), the reduced Planck's constant ($\hbar$), the gravitational constant (G) and the Boltzmann constant (k_B) is called the Planck's unit [2]. This system of units comprises of the three out of the four constants such as the Planck's mass (M_P), Planck's length (l_P), Planck's time (t_P) and Planck's temperature (T_P). The usual base units are chosen to be the mass, length and time omitting temperature since it is redundant. The implication of these four derived constants came to light only after realizing the very need for a `Quantum Theory of Gravity '. Starting from the numerical values of $\hbar$ =1.054571817× 10^{-34} $j.s$, G = 6.674×10^{-11} $m^3.\ Kg^{-1}.\ s^{-2}$, c = 2.99792458× 10^8 $m.s^{-1}$, the four aforesaid constants are computed as follows.

$$M_P = \sqrt{\frac{\hbar c}{G}} = \sqrt{\frac{(1.05\times 10^{-34}\ J.s)\times(3\times 10^8\ m.s^{-1})}{6.67\times 10^{-11}\ m^3.Kg^{-1}.s^{-2}}} = 2.176\times 10^{-8}\ Kg = 2.176\times 10^{-5}\ gm.$$

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = \sqrt{\frac{(1.05\times 10^{-34}\ J.s)\times(6.67\times\ 10^{-11}\ m^3.Kg^{-1}.s^{-2})}{(3\times\ 10^8\ m^1.s^{-1})^3}} \tag{11}$$

$$= 1.616\times 10^{-35}\ m = 1.61610^{-33}\ cm.$$

$$t_P = \sqrt{\frac{\hbar G}{c^5}} = \frac{l_P}{c} = \frac{1.61\times10^{-33}}{3\times 10^{10}}\ s = 5.391 \times 10^{-44}\ s.$$

$$T_P = \frac{M_P c^2}{k_B} = \frac{(2.17\times 10^{-8}\ Kg)\times(9\times 10^{16}\ m^2.s^{-2})}{1.38\times 10^{-23}\ J.K^{-1}} = 1.417 \times 10^{32}\ K.$$

where, the Planck's temperature is computed in view of the mass-energy relationship of Einstein ($E = mc^2$) and the equipartition of energy ($E = \frac{1}{2} k_B T$ for each degree of freedom, $k_B = 1.38\times 10^{-23}\ J.K^{-1}$ being the Boltzmann constant). Note that l_P and t_P are extremely small while T_P is extremely large. On the contrary, $M_P \approx 22\ \mu\ gm$ although is very large in comparison to the masses of fundamental particles, lie very much within the mass range of living orgasms such as female ovum *[Just for the sake of comparison, we remember that the masses of the electron, proton, the earth and the sun are,* $m_e = 9.109 \times 10^{-31}\ Kg$, $m_p = 1.673\times10^{-27}\ Kg$, $M_\oplus = 5.972\times 10^{24}\ Kg$, $M_\odot = 1.988 \times 10^{30}\ Kg$, *respectively while the mass of a female ovum lies within the range* 0.1 - 0.2 *mg*]. However, the Planck's mass actually limits the mass of fundamental particles as well as the minimum size of a black hole. Also note that equation (1) implies $l_P = ct_P$, as it should be, which immediately suggests to verify (starting from the velocity of light in vacuum) that light travels a distance of Planck's length l_P in Planck's time t_P.

In order to realize the implication of these constants, let us refer to astro-particle physics and physical cosmology, in which the Planck's scale is essentially an energy scale at which the quantum effect of gravity becomes significant and the current predictions of physics, such as the standard model, quantum field theory and GTR break down. Therefore, let us first proceed to convert mass to energy. In view of the mass-energy relationship of Einstein $E = mc^2$, one can convert equivalence of 1 *Kg* of mass in terms of the energy (Joule) as,

$$E = 1\ Kg \times 9 \times 10^{16}\ m^2.s^{-2} = 9 \times 10^{16}\ J \tag{12}$$

Thus, it is understood that 1 *Kg* of an object is equivalent to or can produce up to $9\times 10^{16}\ J$ energy. Next, as the amount of energy acquired by an electron (charge $e = 1.6\times10^{-19}\ C$) while moving across a potential difference of 1 *V* is 1 $eV = 1.6\times10^{-19}\ J$, therefore, equivalence of 1 *Kg* mass is,

$$1Kg = 9 \times 10^{16}\ J = \frac{9 \times 10^{16}}{1.6 \times 10^{-19}}\ eV = 5.625 \times 10^{35} eV = 5.625 \times 10^{26}\ GeV \tag{13}$$

and hence, finally in view of (12) and (13) we find,

$$1\ M_P = 2.17 \times 10^{-8}\ Kg = 1.22 \times 10^{19}\ GeV \tag{14}$$

Let us now recall that the electro-weak scale *[the energy scale at which the strengths of electromagnetic interaction and weak interaction are of the same order and therefore indistinguishable]* is around 246 *GeV*, while such symmetry breaks down at around 159.5 *GeV*. Clearly, the energy equivalent to 1 $M_P \approx 22\ \mu\ gm \approx 10^{19}\ GeV$ is exorbitantly large in comparison. The energy scale on the contrary, at which the strength of the electromagnetic interaction, weak interaction and the strong interaction are of equal strength and hence are indistinguishable is given by a `Grand Unified Theory' and is simply called the `GUT' scale, which is around 10^{16} *GeV*. To realize the number, let us refer to the the `Large Hadron Collider' energy scale, which is only around 10^4 *GeV*. Thus, it is not possible to prove GUT from any earth-based accelerator even in the far future. Nonetheless, the stars, supernovae are the natural accelerators for this purpose [*We better also compare luminosity of the sun and the supernova SN1a, which are* $L_\odot = 3.84 \times 10^{24}\ J\ .\ s^{-1}$ and $L_{SN1a} = 10^{41}\ J\ .\ s^{-1}$ *respectively*].

Gravity is the weakest interaction amongst the four interactions. Nonetheless, at 1 $MP \approx 10^{19}\ GeV$ (14), gravitational interaction becomes comparable to the other three and becomes indistinguishable. At this energy scale, the four interactions are supposed to be unified through a viable `Quantum Theory of Gravity', which has been conjectured to be the `Theory of Everything' (TOE). Unfortunately, there is no such theory at hand, despite tremendous effort over several decades. This energy scale corresponds to a length scale l_P , which light can travel in time t_P. Currently, the Planck's scale therefore suggests that present knowledge in physics cannot be extended beyond these values. It is important to remember that GTR suffers from unavoidable singularity problem, i.e., classical gravity theory collapses beyond a length scale l_P, where a quantum theory of gravity is indispensable. The Planck's length scale is also known to be the length scale at which the Compton wavelength of a particle [*Compton wavelength of a particle is given by* $\lambda = \frac{h}{mc}$, *which for an electron is around* $2.4263\times 10^{-13}\ m$. *The inverse of the reduced Compton wavelength* $(\bar{\lambda} = \frac{\hbar}{mc})$ *is a natural representation for the mass on the quantum scale and appears in the relativistic Klein-Gordon equation for a free particle*] is comparable to its Schwarzschild radius *[Schwarzschild solution is the solution of a static spherically symmetric object (such as a star) of Einstein's field equation of GTR. The solution dictates that if a star collapses* *after using up all its fuel, then it would finally become a small object being embedded by a boundary, called the event horizon. The radius of the event horizon* $r_s = \frac{2GM}{c^2}$ *represents the characteristic radius of any quantity of mass. A black-hole formed out of 10 solar mass would have an event horizon radius of around 30 Km only]*. Thus, at present l_P is interpreted as the length scale beyond which one cannot probe. The cosmological Big-Bang singularity likewise dictates that our present knowledge forbids to ask what happened beyond t_P [*Let us mention that in the absence of a viable `Quantum Theory of Gravity', `Quantum Cosmology' has been developed. It is essentially the quantization of cosmological equation, known as the Wheeler-DeWitt (WD) equation. It attempts to probe beyond Planck's time, to get an understanding of the situation. Unfortunately, in Wheeler-de-Witt equation time parameter disappears, which implies that the concept of time does not arise beyond* t_P]. Further, it is not possible to probe the energy scale associated with the Planck's mass M_P and hence no fundamental particle can have mass greater than the Planck's mass M_P. At Planck's temperature, the wavelength of light emitted by thermal radiation reaches Planck's length. The Big-Bang temperature is T_P, and the universe, with our current understanding could not acquire temperature greater than T_P.

4. Natural Unit $\hbar = c = 8\pi G = 1$

Often a geometrized system of units [3] is used as natural units in GTR and also in STR (Special Theory of Relativity), where one sets $c = G = 1$ or more generally $c = 8\pi G = 1$. Particularly, $c = 1$ implies that light in vacuum travels unit distance in unit time and so 1 s is interpreted as one light-second. This sets time and distance on equal footing and so time takes the geometric unit of length. The so-called natural unit sets $\hbar = c = 8\pi G = 1$, in which all the physical parameters become dimensionless. There is apparently no problem to work with the dimensionless parameters, as long as we are in the classical regime, since the reduced Planck's constant $\hbar$ has no role in the classical domain. Let us start with the action (7) in the presence of a scalar field, to investigate what happens in the natural unit $\hbar = c = 8\pi G = M_P^{-1} = 1$,

$$A = \int \left[\frac{1}{2}(R - 2\Lambda) + \beta R^2 + \gamma R^{-1} - \frac{1}{2} g_{\mu\nu}\, \phi^{,\mu} \phi^{,\nu} - V(\phi)\right]\sqrt{-g}\, d^4x \tag{15}$$

We already know that the dimension of $[R]$ and $[\Lambda]$ are $[L^{-2}] = M_P^2 = 1$ and so all the parameters of the action are dimensionless. Nonetheless, it is not much useful, since observation suggests dimensional parameters. Further, in the quantum domain, the perturbative analysis (semiclassical approximation) is carried out either in terms of the reduced Planck's constant ($\hbar$) or the reduced Planck's mass (M_P). Hence, to comply with experimental observations in astro-particle physics as well as with perturbative quantum analysis, it is always suggestive to withdraw the last condition and keep M_P in the action *[Note that, setting reduced Planck's constant* $\hbar = 1$ *has nothing to do with classical GTR. In fact, semi-classical approximation is usually performed following expansion in terms of* $\hbar$. *Hence for this purpose, one can choose* $M_P = 8\pi G = c = 1$, *without fixing* $\hbar$ *to unity. Once classical transition is found to be allowed, in view of an oscillatory wave function, one can fix* $\hbar = 1$ *in the classical domain].*

As for the sake of illustration let us mention that in the background of isotropic and homogeneous Robertson-Walker line-element

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{16}$$

where, *a(t)* is the scale factor, inflation is studied in the natural unit, for a host of potentials (quadratic, quartic, inverse power laws, their combinations and even inverse exponential) in a non-minimally coupled scalar-tensor theory of gravity [4]. The action is given by,

$$A = \int \left[f(\phi)R - \frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi^{,\mu} - V(\phi) - \mathcal{L}_m\right]\sqrt{-g}\, d^4 x \tag{17}$$

where, $f(\phi)$ is the coupling parameter and $\omega(\phi)$ is the variable Brans-Dicke parameter. Applying the following conformal transformation [5]

$$g_{E\mu\nu} = f(\phi) g_{\mu\nu} \tag{18}$$

the above action (17) may be translated to the following Einstein's frame,

$$A = \int \left[R_E - \frac{1}{2}\sigma_{E,\mu}\sigma_E^{,\mu} - V_E\big(\sigma(\phi)\big)\right]\sqrt{-g_E}\, d^4 x \tag{19}$$

where, the subscript `E' stands for Einstein's frame. The transformed scalar field σ and effective potential V_E in the Einstein's frame are related to the parameters of the Jordan frame action (17) through the following expressions,

$$\left(\frac{d\sigma}{d\phi}\right)^2 = 2\frac{\omega(\phi)}{\phi f(\phi)} + 3\frac{f'^2(\phi)}{f^2(\phi)} \text{ and } V_E = \frac{V(\phi)}{f^2(\phi)} \tag{20}$$

$$\ddot{\sigma} + 3\mathrm{H}\dot{\sigma} + k_0 V_E' = 0;\ 3\mathrm{H}^2 = \left(\frac{\dot{a}_E}{a_E}\right)^2 = \frac{1}{2}\dot{\sigma}^2 + V_E \tag{21}$$

Excellent agreement with the latest released Planck's data [6,7] was found [*In the manuscript, by mistake we have used a dimension in the potential parameter, which should be ignored*]. So as mentioned, seemingly there is no problem as such. However, some issues are involved with the natural unit. Firstly, Inflation is a quantum theory of perturbation in the background of classical space-time, which occurred at around 10^{-36} sec. Now, if a quantum theory admits a viable semiclassical (WKB) approximation [*If the semiclassical wavefunction manifests oscillatory behaviour around the classical inflationary solution, which means that the solution is strongly peaked around classical inflationary solutions, then only the quantum theory is supposed to be viable*], then most of the information in connection with inflation may be extracted from the classical field equations only. Therefore, prior to the study of inflation it is required to quantize the theory (i.e., construct modified Wheeler-deWitt equation) and perform semi-classical approximation. In the perturbative analysis of a quantized theory (quantum cosmology, for example), the action (the Hamilton-Jacobi functional S) in the wave function ($\Psi = \Psi_0 e^{\frac{i}{\hbar}S}$) is expanded in terms of $\hbar$ [8,9] or alternatively, in terms of the Planck's mass M_P [10,11] and so either $\hbar$ or M_P must not be fixed to unity, a priori. For further clarification, let us consider the following action [12],

$$A = \int \left[\alpha(\phi)(R - 2\Lambda) + \beta R^2 + \gamma \mathcal{G}^2 - \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi)\right]\sqrt{-g}\, d^4x \tag{22}$$

where, $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ is the Gauss-Bonnet term. In the above, β and γ are constant parameters. In view of de-Sitter solution $a = a_0 e^{\lambda t}$ the parameters $\alpha(\phi)$ and the potential $V(\phi)$ take the following form,

$$\alpha(\phi) = \left[\alpha_0 - \frac{\alpha_1}{\phi} - \frac{\phi^2}{12}\right], V(\phi) = \lambda^2\phi^2 + 12\alpha\lambda^2 - 2\alpha\Lambda + V_0 \tag{23}$$

The quantum counterpart, which is essentially the modified Wheeler-de-Witt equation, reads in the natural unit as [12]

$$i\frac{\partial\Psi}{\partial\sigma} = -\frac{1}{198\left[\gamma x^5 + \beta x \sigma^{\frac{8}{11}}\right]}\left(\frac{\partial^2}{\partial x^2} + \frac{n}{x}\frac{\partial}{\partial x}\right)\Psi - \frac{1}{11x\sigma^{\frac{12}{11}}}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{6i}{11\sigma}\left(\frac{\phi}{6} - \frac{\alpha_1}{\phi^2}\right)\frac{\partial\Psi}{\partial\phi} +$$
$$\frac{6i}{11\sigma}\left(\frac{1}{12} + \frac{\alpha_1}{\phi^3}\right)\Psi + V_e\Psi = \widehat{H}_e\Psi \tag{24}$$

where, $z = a^2, x = \dot{z}, \sigma = z^{\frac{11}{2}} = a^{11}$ and n is the operator ordering index. The effective potential V_e is given by,

$$V_e = \left[\frac{9x}{11\sigma^{\frac{10}{11}}}\left(\frac{\alpha_1^2}{\phi^4} - \frac{\alpha_1}{3\phi} + \frac{\phi^2}{36}\right) - \frac{3x}{11\sigma^{\frac{10}{11}}}\left(\alpha_0 - \frac{\alpha_1}{\phi} - \frac{\phi^2}{12}\right) + \frac{135\gamma x^7}{154\sigma^2}\right.$$
$$\left. + \frac{2}{11x\sigma^{\frac{6}{11}}}\left(-1152\gamma\lambda^8 - 2\frac{V_2}{\phi} - V_1\right)\right] \tag{25}$$

Clearly, one cannot further proceed to find semi-classical (WKB) wave function, in the absence of $\hbar$ and M_P.

Next, all information regarding dimensional analysis is lost and the use of coupling parameters (used for dimensional matching), becomes redundant. Note that Inflation is considered to be a scenario rather than a model for the reason that, different modified and alternative (modified telleparallel gravity) theories of gravitation, different forms of the potential and even curvature induced inflation [13,14], find excellent agreement with the Planck's data [6,7], constraining the parameters of the theory. As mentioned, the parameters are not required any more in the natural unit and so the task becomes very difficult if not impossible and if some model finds agreement, it would be difficult to interpret why universe chose a particular model. Additionally, some physical parameters are measured in appropriate dimensions. For instance, the present value of the Hubble parameter H_0 (measured in the unit Km.s^{-1}.Mpc^{-1} or crossing out the units it may be also be expressed simply in terms of the unit of frequency s^{-1}), the age of the universe t_0 (measured in terms of *Gyr*), so that $H_0 t_0 \approx 1$. It is therefore suggestive to consider the system of units as described in the following section.

5. Dimensions of GTR Parameters in Terms of Planck's Mass

From now on we shall work with the reduced Planck's mass, $M_P = \sqrt{\frac{ch}{8\pi G}} \approx 2.4 \times 10^{18}\, GeV$, just to get rid of $8\pi G$ from the right-hand side of the Einstein's equation (1). Now, if we choose the system of unit $\hbar = c = 1$, while M_P takes the value of reduced Planck's mass, then no such issues discussed above appears. Note that, the reduced Planck's constant, being a universal constant, may always be fixed to $\hbar = 1$, which allows to relate energy unit with the inverse of time unit. Also, one of the fundamental assertions of `Special Theory of Relativity is: the velocity of light in vacuum is isotropic (same in all direction), independent of of the observer and information cannot be carried faster than the speed of light. Hence, one can also fix $c = 1$, which further allows to relate the time unit with the length unit. As a result, all the units may be expressed in terms of different degrees of the energy unit (GeV) or equivalently in terms of different powers of the reduced Planck's mass. This is what we aim at below.

The equivalent distance travelled by light in vacuum in one second is $3\times 10^8\, m$. Hence,

$$1\, s \equiv 3 \times 10^{10}\, cm \tag{26}$$

Further, as $\hbar = \frac{h}{2\pi} = 1.05 \times 10^{-34}\, J.s$, is also set to 1, so

$$1\, J \equiv \frac{1}{1.05} \times 10^{34}\, s^{-1} \equiv \frac{10^{34}}{1.05 \times 3 \times 10^{10}}\, cm^{-1} \tag{27}$$

Hence,

$$1\, GeV = 10^9\, eV = 10^9 \times 1.6 \times 10^{-19} J \equiv \frac{1.6 \times 10^9 \times 10^{-19} \times 10^{34}}{1.05 \times 3 \times 10^{10}}\, cm^{-1}$$
$$= 5.07 \times 10^{13} cm^{-1} \tag{28}$$

Therefore, $1\, M_P = 2.4 \times 10^{18}\, GeV \equiv 1.22 \times 10^{32}\, cm^{-1} = 3.66 \times 10^{42}\, s^{-1}$

and $\quad 1\, s^{-1} = \frac{2.4\times 10^{18}}{3.65\times 10^{42}} = 6.575 \times 10^{-25}\, GeV = 2.732 \times 10^{-43}\, M_P.$

However, we can better relate l_P and t_P with M_P, as follows

$$1\, l_P = 1.61 \times 10^{-33} cm \equiv 6.1854 \times 10^{-4} \times 1.61 \times 10^{-33} M_P^{-1}$$
$$= 9.95 \times 10^{-37} M_P^{-1} \tag{29}$$

$$1\, t_P = 5.37 \times 10^{-44} s \equiv 1.86652 \times 10^{7} \times 5.37 \times 10^{-44} M_P^{-1}$$
$$= 1.0023 \times 10^{-36} M_P^{-1} \tag{30}$$

As an application of the said system of units, let us consider the modified gravitational action in the following form,

$$A = \int \left[\frac{M_P^2}{2}(R - 2\Lambda) + \beta R^2 + \gamma R^{-1} - \frac{1}{2} g_{\mu\nu}\phi^{,\mu}\phi^{,\nu} - V(\phi)\right]\sqrt{-g}\, d^4x \tag{31}$$

Since we know that $[R] = l_P^{-2} = [\Lambda]$, so under the current system of units, $[R] = M_P^2 = [\Lambda]$ and the dimension of the first term in the above action is M_P^4. Hence, the dimensions of $[\beta R^2] = [\gamma R^{-1}] = [\phi^{,\mu}\ \phi^{,\nu}] = [V(\phi)] = M_P^4$ as well. One can easily find that β has to be a dimension-less parameter, while the dimension of $[\gamma] = M_P^6$. Finally, $\phi^{,\mu}\ \phi^{,\nu} = \left(\frac{d\phi}{dx^\mu}\right)\left(\frac{d\phi}{dx^\nu}\right)$, so its dimension is either the dimension of $\left[\left(\frac{d\phi}{dt}\right)^2\right] = \left[\frac{\phi^2}{t_P^2}\right]$ or $\left[\left(\frac{d\phi}{dx^i}\right)\left(\frac{d\phi}{dx^j}\right)\right] = \left[\frac{\phi^2}{l_P^2}\right]$. Hence, the dimension of $[\phi^{,\mu}\ \phi^{,\nu}] = M_P^2\,\phi^2$. Clearly, the dimension of $[\phi] = M_P$. For a crosscheck, remember that the dimension of $[\phi] = M^{1/2} L^{1/2} T^{-1}$ (see table-1), which in the current system of units reads as $[\phi] = M^{\frac{1}{2}} \times M^{-\frac{1}{2}} \times M_P = M_P$. Finally, in the case of a quadratic potential $V(\phi) = V_0\ \phi^2$, V_0 has the dimension of $[V_0] = M_P^2$ (compare with table-1), while for quartic potential $V(\phi) = \lambda\, \phi^4$, λ is dimensionless and finally for an exponential potential such as $V(\phi) = V_0\ e^{\pm\eta\phi}$, the dimension of $[V_0] = M_P^4$ and $[\eta] = [\phi^{-1}] = [M_P^{-1}]$. In general, for a potential given in the form $V(\phi) = V_4\ \phi^n$, the dimension of $[V_4] = M_P^{4-n}$, where n takes takes arbitrary real value. Likewise, for the action [22], we know that $[\alpha] = M_P^2$, hence β is a dimensionless parameter, while $[\gamma] = M_P^{-4}$. In table-2 we present a list of dimensions of the physical parameters in terms of different powers of M_P.

Lagrangian	Lagrangian density	Action	Ricci scalar, Ricci tensor, Cosmological constant	Energy-Momentum Tensor	Scalar field	Potential
$[L]$	$[\mathcal{L}_m]$	$[A]$	$[R], [R_{\mu\nu}], [\Lambda]$	$[T_{\mu\nu}]$	$[\phi]$	$[V(\phi)]$
M_P	M_P^4	Dimension less	M_P^2	M_P^4	M_P	M_P^4

Table 2: Dimensions of the Standard Parameters used in GTR, in Terms of powers of M_P

In table-3, we also present a list of the dimensions of some typical potential parameters.

Quadratic Potential	Cubic Potential	Quartic Potential	Inverse Potential	Exponential Potential
$V(\phi) = V_0\phi^2$	$V(\phi) = V_1\phi^3$	$V(\phi) = \lambda\phi^4$	$V(\phi) = V_2\phi^{-1}$	$V(\phi) = \mathcal{V}_0 e^{\pm\eta\phi}$
L^{-2}	$V_1 = M^{-\frac{1}{2}}L^{-\frac{5}{2}}T$	$\lambda = M^{-1}L^{-3}T^2$	$V_2 = M^{\frac{3}{2}}L^{-\frac{1}{2}}T^{-3}$	$\mathcal{V}_0 = L^{-1}MT^{-2}$
$V_0 = M_P^2$	$V_1 = M_P^1$	λ dimensionless	$V_0 = M_P^5$	$\mathcal{V}_0 = M_P^4$

Table 3: Dimensions of Some Standard Potential Coupling Parameters

We also compute numerical values of some fundamental cosmological parameters in the energy unit. These are enlisted in the following table 4.

Cosmological parameters	Cosmological parameters in terms of powers of energy unit (eV / GeV)
G: Newton's Gravitational constant.	$G = \frac{1}{8\pi M_P^2} = \frac{1}{8\pi \times 2.4^2 \times 10^{36}} = 6.91 \times 10^{-39}\ GeV^{-2}$
H_0: Hubble parameter, present value for ($h = 0.7$).	$100h\ Km.s^{-1}Mpc^{-1} = \frac{100h}{3 \times 10^5} Mpc^{-1} = \frac{h}{3000} Mpc^{-1} = \frac{0.7}{3000 \times (3.086 \times 10^{24}) \times (5.068 \times 10^{13})} = 1.492 \times 10^{-42}\ GeV = 6.22 \times 10^{-61}\ M_P$
H_e: Hubble parameter at Big Bang	Inflation lasted for $\Delta t \approx 10^{-36} s\ (10^{-40\pm2} - 10^{-36\pm2})\ s$, during which universe expanded 60 e – fold times, i.e., $\Delta N \approx 60$. So $H_e = \dot{N} = \frac{dN}{dt} = \frac{60}{10^{-36}s} = 60 \times 10^{36} \times 2.732 \times 10^{-43} = 1.64 \times 10^{-5}\ M_P$

ρ_Λ	$H^2 = \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho_\Lambda, ie., \rho_\Lambda = \frac{3H^2}{8\pi G} = 3 \times M_P^2 \times H^2$ Therefore, Present value of vacuum energy density is $\rho_\Lambda = 3 \times (2.4 \times 10^{18})^2 \times (1.492 \times 10^{-42})^2\ GeV^2 = 3.8 \times 10^{-47}\ GeV^4$
$\rho_{\rm vac}$ The sum of zero-point energies of vibrational modes of all the quantum fields.	$\langle T_{00}\rangle_{vac} = \rho_{vac} = \int_0^{\rho_{max}} K^3\, dK = 10^{73}\ GeV^4.$ A naive calculation: Frequency at Grand Unified (GUT) scale is $\nu_{GUT} \approx 2.4 \times 10^{39}\ Hz.$ $\rho_{Gut} = \int_{\nu_{min}}^{\nu_{GUT}} \frac{8\pi\nu^2}{c^3} \times \frac{h\nu}{2} d\nu = 2.56 \times 10^{99} J = \left(\frac{2.56\times10^{99}}{16\times10^{-10}}\right) \times \frac{1}{(5.07\times10^{11})^3}$ So, $\rho_{GUT} \approx 3.15 \times 10^{74}\ GeV.$ Thus, $\frac{\rho_{\rm vac}}{\rho_\Lambda} \approx 10^{-120}$
R_e: Value of Ricci scalar on earth	Einstein's GTR implies: $R = \frac{8\pi G}{c^4} T = \frac{8\pi G}{c^4}(\rho c^2 - 3p)$ for perfect fluid. For objects like the sun, the earth one can neglect pressure. So in the present unit, $R = \frac{\rho}{M_P^2}$. So, $R_e = \frac{5.5}{M_P^2}\ gm.cm^{-3}$. Using relations (13) and (28) one can compute $R_e \approx 10^{-35}\ eV^2$.

Table 4: Some Important Physical Parameters in Different Powers of Energy Unit

Energy-Tempaerature Relation

Finally, we present temperature-energy relation

$1.602\times 10^{-19} J = 1\ eV = k\,T = 1.38 \times 10^{-23} J.K^{-1} \times T.$

Therefore, temperature corresponding to 1 eV is,

$\mathrm{T} = \frac{1.602}{1.38} \times 10^4\ K = 1.16087 \times 10^4\ K = 11609\ K.$

Conversely, 1 $K = 8.614 \times 10^{-5}\ eV$.

Temperature corresponding to the ionizing potential of hydrogen and the present cosmic microwave background temperature are, 13.6 eV = 157878 K and 2.7K = 2.3 × 10^{-4} eV = 10^{-3} eV respectively.

Universe was reheated after graceful exit from inflation due to rapid oscillation of the scalar field that drives inflation, which results in particle production. In this process, hot big-bang resurrects with temperature equivalent to 10^9 GeV = 10^{18} eV = 1.16× 10^{22} K.

Finally, for the sake of completeness, let us mention that in the recent years teleparallel gravity theories are in the limelight. In metric teleparallel theory, which is also dubbed as gravity with torsion, the torsion scalar T plays the role of the Ricci scalar R. In symmetric teleparallel theory, on the contrary, the non-metricity scalar Q plays the role of R. Thus, both T and Q have the dimension of R.

6. Concluding Remarks

In this manuscript, we have explored virtues and some problems associated with different system of units used in high energy physics, such as, GTR, Astroparticle physics, Physical cosmology etc. We think that the most comfortable system of units requires a subtle relaxation of Planck's natural system of units, by choosing $\hbar = c = 1$, while the reduced Planck's mass, viz. $M_P^2 = (8\pi G)^{-1}$ is not set to unity. It allows to express all the physical parameters in terms of different powers of M_P which helps to compare theoretically obtained parameters with observational data. Further, in the quantum domain, one can perform perturbative analysis by expanding the Hamilton-Jacobi function in terms of different powers of M_P. However, if one finds discomfort, it is then suggestive not to fix $\hbar$ to unity, while working in the quantum domain. Once, classical transition is established under a suitable (Oscillatory wave function) semi-classical approximation, one can then fix $\hbar = 1$ and do the computations in the classical regime, so that all the parameters are expressed in terms of M_P yet again. In this process, all the stuffs get easier to set and compare. For example, in the chaotic inflationary scenario, the scalar field slow rolls from a value $\phi_i > M_P$. The inflation ends as $\phi_f < M_P$, where the suffix i and f stand for the initial and the final values of ϕ. The scale of inflation given by the Hubble parameter H_* should be of the order of $H_* \approx 10^{-5} M_P$ and so on.

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