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Short Article

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# Cauchy-Riemann Equations for Treons

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**Abstract**
*We explored an extension of the Cauchy-Riemann equations to the algebra of treons, recently described by Alejandro Bermejo, whose elements are ordered 3-tuples. We leveraged the isomorphism between the algebra of treons and algebra B, and deduced the Cauchy-Riemann equations for the algebra of treons, establishing the necessary conditions for analyticity in this algebraic structure. This work significantly broadened our horizons in complex analysis and introduced new possibilities for applications across various fields of advanced mathematics.*

**Keywords:** Cauchy-Riemann Equations, Treonic Number, Treons, Algebra B, Bermejo's Algebra

## 1. Introduction

The algebra $B$ is a recently described algebraic structure by Alejandro Bermejo, whose elements are ordered 3-tuples $(x_1, x_2, x_3)$ [1]. This algebra is isomorphic to another algebra whose elements, termed treons by Bermejo, are represented in the form $x_1 + x_2 i + x_3 j$, where $i^2 = j^2 = -id$, with id denoting the identity element of the algebra where treons are defined [1,2]. The structure of these treons allows for an extension of the concept of complex numbers to a system with new possibilities in the study of complex analysis.

The Cauchy-Riemann equations are a set of necessary conditions that functions of a complex variable must satisfy to be holomorphic (analytic) in a domain [3,4]. These equations are fundamental in complex analysis because the differentiability of complex functions is strictly related to the fulfillment of these equations [3-5].

We extend the framework of the Cauchy-Riemann equations to the algebra of treons. Utilizing the isomorphism between algebra B and the algebra of treons, we propose a derivation of the Cauchy- Riemann equations adapted to this new algebraic structure [2]. This development not only broadens the boundaries of complex analysis but also establishes a foundation for future research in advanced algebras and their applications. The derivation of these equations in the context of treons is crucial for understanding and leveraging the analytic properties in this algebraic structure.

## 2. Derivation

Due to the isomorphism between the algebra $B$ and the algebra of treons, $X$ [2], we can establish the equivalence:

$$(x_1, x_2, x_3) \equiv x_1 \mathrm{id} + x_2 i + x_3 j,$$

where $(x_1, x_2, x_3) \in B$ and $x_1 + x_2 i + x_3 j \in X$. For simplicity, we assume id $\equiv 1$.

Let an arbitrary treon be $x \equiv x_1 + x_2 i + x_3 j$ and let $f(x)$ be a mapping, we have:

$$f((x_1, x_2, x_3)) = f(x_1 + x_2 i + x_3 j).$$

Let $u((x_1, x_2, x_3))$, $v((x_1, x_2, x_3))$, and $w((x_1, x_2, x_3))$ be three mappings such that:

$$f(x_1 + x_2 i + x_3 j) = u(x_1, x_2, x_3) + iv(x_1, x_2, x_3) + jw(x_1, x_2, x_3).$$

where we simplify the notation of two parentheses as follows: $\Phi((x_1, x_2, x_3)) \equiv \Phi(x_1, x_2, x_3)$.

Let a fixed point $x_0 \equiv (x_{1_0}, x_{2_0}, x_{3_0}) = x_{1_0} + x_{2_0} i + x_{3_0} j$.

With all this, we evaluate each of the limits for the corresponding directions of the three components of the treon $x$.

### 2.1 Variation in the First Component of a Treon

Let there be an arbitrary increment in the direction of the first component, $\Delta x_1$, with respect to the fixed point $x_0$. Then, the limit of $f(x)$ as $\Delta x_1 \to 0$:

$$\lim_{\Delta x_1 \to 0} \frac{f(x_0 + \Delta x_1) - f(x_0)}{\Delta x_1} = \frac{\partial}{\partial x_1} f(x_0),$$

where $x = x_0 + \Delta x_1$, thus $\Delta x_1 = x - x_0$. Then:

$$\frac{\partial}{\partial x_1} f(x_0) = \lim_{\Delta x_1 \to 0} \frac{f(x_{1_0} + \Delta x_1, x_{2_0}, x_{3_0}) - f(x_{1_0}, x_{2_0}, x_{3_0})}{\Delta x_1},$$

which can be rewritten as:

$$\begin{aligned}
\frac{\partial}{\partial x_1} f(x_0) = \lim_{\Delta x_1 \to 0} &\left[ \frac{u(x_{1_0} + \Delta x_1, x_{2_0}, x_{3_0}) + iv(x_{1_0} + \Delta x_1, x_{2_0}, x_{3_0}) + jw(x_{1_0} + \Delta x_1, x_{2_0}, x_{3_0})}{\Delta x_1} \right. \\
&\left. - \frac{u(x_{1_0}, x_{2_0}, x_{3_0}) + iv(x_{1_0}, x_{2_0}, x_{3_0}) + jw(x_{1_0}, x_{2_0}, x_{3_0})}{\Delta x_1} \right] \\
= \lim_{\Delta x_1 \to 0} &\left( \frac{u(x_{1_0} + \Delta x_1, x_{2_0}, x_{3_0}) - u(x_{1_0}, x_{2_0}, x_{3_0})}{\Delta x_1} \right) \\
&+ i \lim_{\Delta x_1 \to 0} \left( \frac{v(x_{1_0} + \Delta x_1, x_{2_0}, x_{3_0}) - v(x_{1_0}, x_{2_0}, x_{3_0})}{\Delta x_1} \right) \\
&+ j \lim_{\Delta x_1 \to 0} \left( \frac{w(x_{1_0} + \Delta x_1, x_{2_0}, x_{3_0}) - w(x_{1_0}, x_{2_0}, x_{3_0})}{\Delta x_1} \right) \\
= &\left[ \frac{\partial u}{\partial x_1} + i \frac{\partial v}{\partial x_1} + j \frac{\partial w}{\partial x_1} \right]_{x_0}.
\end{aligned}$$

### 2.2 Variation in the First Imaginary Component of a Treon

Let there be an arbitrary increment in the direction of the first imaginary component, $i\Delta x_2 \equiv x2_{\text{final}}\, i - x2_{\text{initial}}\, i$, with respect to the fixed point $x_0$. Then, the limit of $f(x)$ as $i\Delta x_2 \to 0$ is:

$$\lim_{\Delta x_2 \to 0} \frac{f(x_0 + i\Delta x_2) - f(x_0)}{i\Delta x_2} = \frac{\partial}{i\partial x_2} f(x_0),$$

where $x = x_0 + i\Delta x_2$, thus $i\Delta x_2 = x - x_0$. Then:

$$\frac{\partial}{i\partial x_2} f(x_0) = \lim_{\Delta x_2 \to 0} \frac{f(x_{1_0}, x_{2_0} + \Delta x_2, x_{3_0}) - f(x_{1_0}, x_{2_0}, x_{3_0})}{i\Delta x_2},$$

which can be rewritten as:

$$\begin{aligned}
\frac{\partial}{i\partial x_2} f(x_0) = \lim_{\Delta x_2 \to 0} &\left[ \frac{u(x_{1_0}, x_{2_0} + \Delta x_2, x_{3_0}) + iv(x_{1_0}, x_{2_0} + \Delta x_2, x_{3_0}) + jw(x_{1_0}, x_{2_0} + \Delta x_2, x_{3_0})}{i\Delta x_2} \right. \\
&\left. - \frac{u(x_{1_0}, x_{2_0}, x_{3_0}) + iv(x_{1_0}, x_{2_0}, x_{3_0}) + jw(x_{1_0}, x_{2_0}, x_{3_0})}{i\Delta x_2} \right] \\
= \lim_{\Delta x_2 \to 0} &\left( \frac{u(x_{1_0}, x_{2_0} + \Delta x_2, x_{3_0}) - u(x_{1_0}, x_{2_0}, x_{3_0})}{i\Delta x_2} \right) \\
&+ i \lim_{\Delta x_2 \to 0} \left( \frac{v(x_{1_0}, x_{2_0} + \Delta x_2, x_{3_0}) - v(x_{1_0}, x_{2_0}, x_{3_0})}{i\Delta x_2} \right) \\
&+ j \lim_{\Delta x_2 \to 0} \left( \frac{w(x_{1_0}, x_{2_0} + \Delta x_2, x_{3_0}) - w(x_{1_0}, x_{2_0}, x_{3_0})}{i\Delta x_2} \right) \\
= &\left[ \frac{\partial u}{i\partial x_2} + i \frac{\partial v}{i\partial x_2} + j \frac{\partial w}{i\partial x_2} \right]_{x_0}
\end{aligned}$$

$$= \left[\frac{\partial v}{\partial x_2} - i\frac{\partial u}{\partial x_2} + j\frac{\partial w}{i\partial x_2}\right]_{x_0}.$$

### 2.3 Variation in the Second Imaginary Component of a Treon

Let there be an arbitrary increment in the direction of the second imaginary component, $j\Delta x_3$, with respect to the fixed point $x_0$. Then, the limit of $f(x)$ as $j\Delta x_3 \to 0$ is:

$$\lim_{\Delta x_3 \to 0} \frac{f(x_0 + j\Delta x_3) - f(x_0)}{j\Delta x_3} = \frac{\partial}{j\partial x_3} f(x_0),$$

where $x = x_0 + j\Delta x_3$, thus $j\Delta x_3 = x - x_0$. Then:

$$\frac{\partial}{j\partial x_3} f(x_0) = \lim_{\Delta x_3 \to 0} \frac{f(x_{1_0}, x_{2_0}, x_{3_0} + j\Delta x_3) - f(x_{1_0}, x_{2_0}, x_{3_0})}{j\Delta x_3},$$

which can be rewritten as:

$$\begin{aligned}
\frac{\partial}{j\partial x_3} f(x_0) &= \lim_{\Delta x_3 \to 0} \Bigg[\frac{u(x_{1_0}, x_{2_0}, x_{3_0} + j\Delta x_3) + iv(x_{1_0}, x_{2_0}, x_{3_0} + j\Delta x_3) + jw(x_{1_0}, x_{2_0}, x_{3_0} + j\Delta x_3)}{j\Delta x_3} \\
&\quad - \frac{u(x_{1_0}, x_{2_0}, x_{3_0}) + iv(x_{1_0}, x_{2_0}, x_{3_0}) + jw(x_{1_0}, x_{2_0}, x_{3_0})}{j\Delta x_3}\Bigg] \\
&= \lim_{\Delta x_3 \to 0} \left(\frac{u(x_{1_0}, x_{2_0}, x_{3_0} + j\Delta x_3) - u(x_{1_0}, x_{2_0}, x_{3_0})}{j\Delta x_3}\right) \\
&\quad + i \lim_{\Delta x_3 \to 0} \left(\frac{v(x_{1_0}, x_{2_0}, x_{3_0} + j\Delta x_3) - v(x_{1_0}, x_{2_0}, x_{3_0})}{j\Delta x_3}\right) \\
&\quad + j \lim_{\Delta x_3 \to 0} \left(\frac{w(x_{1_0}, x_{2_0}, x_{3_0} + j\Delta x_3) - w(x_{1_0}, x_{2_0}, x_{3_0})}{j\Delta x_3}\right) \\
&= \left[\frac{\partial u}{j\partial x_3} + i\frac{\partial v}{j\partial x_3} + j\frac{\partial w}{j\partial x_3}\right]_{x_0} \\
&= \left[\frac{\partial w}{\partial x_3} + i\frac{\partial v}{j\partial x_3} - j\frac{\partial u}{\partial x_3}\right]_{x_0}.
\end{aligned}$$

Therefore, we have three equations:

$$\begin{aligned}
\frac{\partial}{\partial x_1} f(x_0) &= \left[\frac{\partial u}{\partial x_1} + i\frac{\partial v}{\partial x_1} + j\frac{\partial w}{\partial x_1}\right]_{x_0}, \\
\frac{\partial}{i\partial x_2} f(x_0) &= \left[\frac{\partial v}{\partial x_2} - i\frac{\partial u}{\partial x_2} + j\frac{\partial w}{i\partial x_2}\right]_{x_0}, \\
\frac{\partial}{j\partial x_3} f(x_0) &= \left[\frac{\partial w}{\partial x_3} + i\frac{\partial v}{j\partial x_3} - j\frac{\partial u}{\partial x_3}\right]_{x_0}.
\end{aligned}$$

For the derivative to exist at the fixed point, the limit must be the same regardless of the direction in which it is evaluated. Moreover, one treon is equal to another treon if their respective components are equal [1,2]. According to this, we have:

$$\left[\frac{\partial u}{\partial x_1} + i\frac{\partial v}{\partial x_1} + j\frac{\partial w}{\partial x_1}\right]_{x_0} = \left[\frac{\partial v}{\partial x_2} - i\frac{\partial u}{\partial x_2} + j\frac{\partial w}{i\partial x_2}\right]_{x_0} = \left[\frac{\partial w}{\partial x_3} + i\frac{\partial v}{j\partial x_3} - j\frac{\partial u}{\partial x_3}\right]_{x_0}.$$

Comparing component by component, we have:

- If $\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}$ and $i\frac{\partial v}{\partial x_1} = -i\frac{\partial u}{\partial x_2}$, then $j\frac{\partial w}{\partial x_1} = j\frac{\partial w}{i\partial x_2}$.
- If $\frac{\partial u}{\partial x_1} = \frac{\partial w}{\partial x_3}$ and $j\frac{\partial w}{\partial x_1} = -j\frac{\partial u}{\partial x_3}$, then $i\frac{\partial v}{\partial x_1} = i\frac{\partial v}{j\partial x_3}$.

We are left with the following equations:

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2} = \frac{\partial w}{\partial x_3}, \quad i\frac{\partial v}{\partial x_1} = -i\frac{\partial u}{\partial x_2}, \quad j\frac{\partial w}{\partial x_1} = -j\frac{\partial u}{\partial x_3},$$

which is the corresponding Cauchy-Riemann equations for the algebra of treons.
Therefore, these are the required conditions for an arbitrary function with a treon domain to be" holomorphic" in some subspace of the treon space.

## 3. Conclusions

We derived the Cauchy-Riemann equations for the algebra of treons, extending the concept of holo- morphism to this algebra. Utilizing the isomorphism between the algebra B and the algebra of treons, we demonstrated that treonic functions defined in this context must satisfy equations analogous to the classical Cauchy-Riemann equations.

Our resulting equations represent a natural extension of the Cauchy-Riemann equations to the realm of treons. This result not only expands the tools available in complex analysis but also introduces a new paradigm for research in advanced algebra.

Our Cauchy-Riemann equations for treons represent a significant advancement in the field of algebraic analysis, establishing a solid foundation for the study and application of algebraic structures in various areas of modern mathematics.

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