ISSN: 2834-7706

Research Article

Journal of Mathematical Techniques and Computational Mathematics

Another Proof of Free Ribbon Lemma

Akio Kawauchi*

Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University Sugimoto, Japan

Corresponding Author
Akio Kawauchi, Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University Sugimoto, Japan.

Submitted: 2024, Jul 22; **Accepted:** 2024, Aug 07; **Published:** 2024, Aug 26

Citation: Kawauchi, A. (2024). Another Proof of Free Ribbon Lemma. *J Math Techniques Comput Math, 3*(8), 01-03.

Abstract
Free ribbon lemma that every free sphere-link in the 4-sphere is a ribbon sphere-link is shown in an earlier paper by the author. In this paper, another proof of this lemma is given.

Keywords: Free Ribbon Lemma, Wirtinger Presentation, Ribbon Sphere-Link

Mathematics Subject Classification 2010: Primary 57N13; Secondary 57Q45

1. Introduction

A *surface-link* is a closed oriented (possibly, disconnected) surface L smoothly embedded in the 4-sphere S^4. When L is connected, L is called a *surface-knot.* If L consists of 2-spheres L_i $(i = 1, 2, \dots, n)$, then L is called a *sphere-link* (or an S^2-link) of n components. It is shown that a surface-link L is a trivial surface-link, i.e., bounds disjoint handlebodies in S^4 if $\pi_1(S^4 \setminus L, x_0)$ is a meridian-based free group [4-6]. A surface-link L is *ribbon* if L is obtained from a trivial S^2-link O in S^4 by surgery along smoothly embedded disjoint 1-handles on O. A surface-link L in the 4-sphere S^4 is *free* if the fundamental group $\pi_1(S^4 \setminus L, x_0)$ is a (not necessarily meridian-based) free group. The free ribbon lemma is the following theorem.

Theorem

Every free S^2-link is a ribbon S^2-link.

This theorem is a basic result concerning Whitehead aspherical conjecture and classical Poincaré conjecture and the proof is done as an appendix [8-10].

At present, it appears unknown whether or not every free surface-link is a ribbon surface-link. In this paper, another proof of this theorem is given as follows.

Proof of Theorem

Let L_i $(i = 1, 2, \dots, n)$ be the components of a free S^2-link L. Let x_i $(i = 1, 2, \dots, n)$ be a basis of the free fundamental group $G = \pi_1(S^4 \setminus L, x_0)$. Let y_i be a meridian element of L_i in G, so that y_i $(i = 1, 2, \dots, n)$ are a meridian system of G. By Nielsen transformations, y_i is equal to x_i modulo the commutator subgroup $[G,G]$ of G. It is known that the group G is isomorphic to a group G^P with Wirtinger presentation

$$P =< y_{ij}\,(1 \le j \le m_i,\, 1 \le i \le n) |\, r_{ij}\,(2 \le j \le m_i + s_i,\, 1 \le i \le n) >$$

such that $y_{i1} = y_i$ $(i = 1, 2, \dots, n)$ and the relators r_{ij} $(j = 2, 3, \dots, m_i + s_i$, $i = 1, 2, \dots, n)$ are given by $r_{ij} : y_{ij} = w_{ij} y_{i1} w_{ij}^{-1}$ for j with $2 \le j \le m_i$, $1 \le i \le n$, and $r_{ij} : y_{i1} = w_{ij} y_{i1} w_{ij}^{-1}$ for j with $m_i + 1 \le j \le m_i + s_i$, $1 \le i \le n$, where w_{ij} $(j = 2, 3, \dots, m_i + s_i$, $i = 1, 2, \dots, n)$ are words in the letters y_{ij} $(j = 1, 2, \dots, m_i$, $i = 1, 2, \dots, n)$. This result is obtained from Yajima [13] because G has a weight system y_i $(i = 1, 2, \dots, n)$, $H_1(G; Z) \cong Z^n$ and $H_2(G; Z) = 0$. It is observed that this result can be also obtained by an alternative geometric method using a normal form of a surface-link in R^4 [11]. In fact, put the S^2-link L in a normal form of in the 4-space R^4 with $L[0] = L \cap R^3[0]$ a middle cross-sectional link and calculate the fundamental groups $\pi_1(R^3[0, +\infty) \setminus L \cap R^3[0, +\infty), x_0)$ and $\pi_1(R^3(-\infty, 0] \setminus L \cap R^3(-\infty, 0], x_0)$ with Wirtinger presentations starting from the fundamental group $\pi_1(R^3[0] \setminus L[, 0], x_0)$ with a Wirtinger presentation to obtain the group G with a Wirtinger presentation by van Kampen theorem [1-3]. By fixing an isomorphism $G^P \to G$, regard the generators y_{ij} $(j = 1, 2, \dots, m_i$, $i = 1, 2, \dots, n)$ of P as fixed words in the basis x_i $(i = 1, 2, \dots, n)$ of G. Then the relator $y_{i1} = w_{ij} y_{i1} w_{ij}^{-1}$ for every i

and j with $m_i + 1 \leq j \leq m_i + s_i$ can be written as $y_{i1} = a_{ij}^{\ u(i,j)}$ and $w_{ij} = a_{ij}^{\ v(i,j)}$ for a word a_{ij} in x_i ($i = 1, 2, \ldots, n$) and some integers $u(i, j)$, $v(i, j)$ because any nontrivial abelian subgroup of a free group is an infinite cyclic group. The elements $y_i = y_{i1}$ ($i = 1, 2, \ldots, n$) form the same abelian basis as x_i ($i = 1, 2, \ldots, n$) in the free abelian group $G/[G, G]$, so that $u(i, j) = \pm 1$ for every i and j. Thus, $w_{ij} = y_{i1}^{\ u(i,j)v(i,j)}$ for every i and j with $m_i + 1 \leq j \leq m_i + s_i$, which means that the relators r_{ij}: $y_{i1} = w_{ij} y_{i1} w_{ij}^{-1}$ ($m_i + 1 \leq j \leq m_i + s_i$) are identity relations in the free group $< y_{ij}$ ($1 \leq j \leq m_i$, $1 \leq i \leq n$) $>$. Thus, the Wirtinger presentation P is equivalent to the Wirtinger presentation

$$R =< y_{ij}\,(1 \leq j \leq m_i,\, 1 \leq i \leq n) |\, r_{ij}\,(2 \leq j \leq m_i,\, 1 \leq i \leq n) >$$

with $y_{i1} = y_i$ ($i = 1, 2, \ldots, n$) and the relators r_{ij} ($2 \leq j \leq m_i$, $1 \leq i \leq n$) given by r_{ij} : $y_{ij} = w_{ij} y_{i1} w_{ij}^{-1}$ ($2 \leq j \leq m_i$, $1 \leq i \leq n$). By Yajima's construction there is a ribbon S^2-link L^R with the fundamental group $G^R = \pi_1(S^4 \setminus L^R, x_0)$ of the Wirtinger presentation R which is isomorphic to G by an isomorphism $G^R \to G$ sending a meridian element y^R_i of the ith component L^R_i of L^R to the meridian element y_i of L_i in G for every i ($i = 1, 2, \ldots, n$) and a basis x^R_i ($i = 1, 2, \ldots, n$) of G^R to the basis x_i ($i = 1, 2, \ldots, n$) of G [2,3,12]. Let Y^R and Y be the 4D manifolds (both diffeomorphic to the n-fold connected sum of $S^1 \times S^3$) obtained from S^4 by surgeries along L^R and L, respectively, and ℓ^R_i ($i = 1, 2, \ldots, n$) and ℓ_i ($i = 1, 2, \ldots, n$) the loop systems obtained from L^R_i ($i = 1, 2, \ldots, n$) and L_i ($i = 1, 2, \ldots, n$), respectively. By [8], there is an orientation-preserving diffeomorphism $f : Y^R \to Y$ sending the loop system ℓ^R_i ($i = 1, 2, \ldots, n$) to the loop system ℓ_i ($i = 1, 2, \ldots, n$). Note that this result is obtained from the smooth unknotting conjecture for S^2-knots and the 4D smooth Poincaré conjecture [4-7]. By the back surgeries from Y^R to S^4 along ℓ^R_i ($i = 1, 2, \ldots, n$) and from Y to S^4 along ℓ_i ($i = 1, 2, \ldots, n$), this diffeomorphism f induces an orientation-preserving diffeomorphism f' : $S^4 \to S^4$ sending L^R to L. Thus, the S^2-link L is a ribbon S^2-link. This completes the proof of Theorem.

In the proof of Theorem, the ribbon S^2-link L^R is called a *ribbon presentation* of the free S^2-link L. The following corollary is obtained from the proof of Theorem.

Corollary
Let L be a free S^2-link in the 4-sphere S^4 containing a free S^2-link K as a sublink. For any ribbon presentation of K^R of K, there is a ribbon presentation L^R of L containing K^R as a sublink.

Proof of Corollary
The ribbon presentation of K^R of K is in a normal form. Thus, the result is obtained from the observation that a normal form of L is taken to contain K^R as a sublink [11].

Acknowledgements
This work was partly supported by JSPS KAKENHI Grant Number JP21H00978 and MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165.

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