

Research Article 1.2 Conditions on a C1-function defined on a Real Banach Space 3

Current Research in Statistics & Mathematics 2π Critical Points 7 1.2 Conditions on a C1-function defined on a Real Banach Space 3

An Exposition on Critical Point Theory with Applications 3 Weak Topology on Banach Spaces 13 **An Exposition on Critical Point Theory with Applications** 1.2 Conditions on a C1-function defined on a Real Banach Space 3 An exposition on Critical Point-I neory with Ap

Subham De* 3 Command De

Department of Mathematics, Indian Institute of Technology, *Delhi, India* 3.1 Weak Convergence . 13 Δ Delhi, India Δ Banach Subash Delhi, India Banach Spaces 13

*** Corresponding Author**

itics, Indian Institute of Technology,
Subham De, Department of Mathematics, Indian Institute of Technology, Delhi, India. Δ Delhi, India. Δ Delhi, India. 3.1 Weak Convergence . 13

Submitted: 2024, Oct 18; Accepted: 2024, Nov 25; Published: 2024, Dec 04 3.33 applications 202π , Oct 10, Accepted. 202π , NOV 23 , Tubished $\sum_{i=1}^{n} S_{i} = \sum_{i=1}^{n} S$

Citation: De, S. (2024). An Exposition on Critical Point Theory with Applications. Curr Res Stat Math, 3(3), 01-31. Crtation: De, S. (2024) . All Exposition on Critical Foliit Theory with Applic.

Abstract 3.3.3 Application in Non-Linear PDEs . 19 $\Delta\text{bstraet}$

Critical Point Theory plays a pivotal role in the study of Partial Differential Equations (PDEs), particularly in investigating
 Cruical Folm Theory plays a pivolal role in the stuay of Farual Differential Equations (FDEs), particularly in investigating
the existence, uniqueness, and multiplicity of weak solutions to elliptic PDEs under specified bo offers a concise survey of key concepts, including differentiation on Banach spaces, the analysis of maxima and minima, and their applications to PDEs. Additionally, we explore the construction of weak topologies on Banach spaces before introducing the Variational Principle and its relevance to solving PDEs. In the final section, we focus on results pertaining to the existence of weak solutions for Dirichlet boundary value problems under specific conditions. A highlight of this work is the proof of Rabinowitz's
Saddle Daint Theorem wis the Provence Desponse websel. Programb we internated in analyzing th Saddle Point Theorem via the Brouwer Degree method. Researchers interested in exploring the themes covered in this paper will
and the Brown will be a search to the San William Sales will be a search to the Brown will be a Saddle Point Theorem via the Brouwer Degree method. Researchers
find the reference section to be a valuable resource for further study. μ existence, uniqueness, and multiplicity of weak solutions to etaplic P . $\frac{1}{2}$ and the Critical Section to the Chinamical Critical Point $\frac{1}{2}$ and $\frac{1}{2}$ similarly $\frac{1}{2}$

> Keywords: Critical Point, Saddle Point, Banach Spaces, Weak Convergence, Variational Principle, Weak Solution, Mountain Pass Theorem, Rabinowitz Saddle Point Theorem, Brouwer Degree \mathcal{A} priori given a partial differential equation on a bounded domain, be it linear, we note it linear, we no ore

1 Introduction \mathbf{m} solutions which also happens to be critica points of certain functionals o $1.1 \cdot 10^{-1}$

1.1 Preliminaries

A priori given a *partial differential* equation on a bounded domain, be it *linear* or *non-linear*, we might end up obtaining solutions which also happens to be *critica points* of certain functionals defined on some appropriate Sobolev Space. Suppose, we consider the following boundary value problem on some bounded domain $\Omega \subset \mathbb{R}^n$ as follows: $\frac{1}{2}$ and uppens to be critical points of certain functionals defined on solite application $\ddot{}$

$$
\begin{cases}\n-\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega\n\end{cases}
$$
\n(1.1)

Where, $\partial\Omega$ denotes the *boundary* of Ω in \mathbb{R}^n . A priori from the boundary condition, it only suffices to search for weak solutions of (1.1) in the Sobolev Space $H_0^1(\Omega)$, the later assertion follows from the fact that, not all functions in $H_0^1(\Omega)$ is smooth (i.e. C^2). $\mathbf{I} = \mathbf{0} \times \mathbf{N}$ \mathbb{R}^2 of \mathbb{R}^2 is defined to be a solution of \mathbb{R}^2

finition 1.1.1. (Weak Solution) A function $u \in H_0^1(\Omega)$ is defined to be a weak solution of (1.1) if follows from the fact that, not all functions in H¹ **Definition 1.1.1.** (Weak Solution) A function $u \in H_0^1(\Omega)$ is defined to be a weak solution of (1.1) if, $(1,1)$

$$
\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi \qquad \forall \ \phi \in H_0^1(\Omega)
$$
\n(1.2)

u is a chassical solution, i.e., in other words, when, $u \in C$ (sz) (C (sz) and satisfies (1.1) point-wise, then, we can
consider the later (1.1) by Lee J integrating by gards obtain (1.2) by multiplying (1.1) by ϕ and integrating by parts. In case when u is a *classical solution*, i.e., in other words, when, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and satisfies (1.1) point-wise, then, we can indeed In case when u is a classical solution, i.e., i.e., in other words, when, u α (1.1) point-wise, then, we can independent obtained obtained obtain $u \in C$ (22) $C(22)$ and satisfies (1.1) by multiplying (1.1) by $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1$

0 (Q) −
| 2 (Q) −→ R as, R a

ł

Ξ

Remark 1.1.1. (1.2) is valid only when, $f \in H^{-1}(\Omega) = (H_0^1(\Omega))^*$. $R = \frac{1}{2}$

^J(u) := ¹

ł

Ξ

Define a functional, $J: H_0^1(\Omega) \longrightarrow \mathbb{R}$ as,

Curr Res Stat Math, 2024

$$
J(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u \tag{1.3}
$$

ced that, J is in fact a C¹-function, and, $J'(u)$: $H_0^+(\Omega) \to \mathbb{R}$ is a bounded linear functional having the following ex It can be deduced that, J is in fact a C¹-function, and, $J'(u)$: $H_0^1(\Omega) \to \mathbb{R}$ is a bounded linear functional having the following expression,

0 (Q) −
| 2 (Q) −→ R as, R a

$$
J'(u)\phi = \int_{\Omega} \nabla u \nabla \phi - \int_{\Omega} f \phi \qquad \forall \phi \in H_0^1(\Omega)
$$
 (1.4)

n infer that, $u \in H_0^1(\Omega)$ is a weak solution of $(1.1) \Leftrightarrow J'(u_0) = 0$. Hence, we can infer that, $u \in H_0^1(\Omega)$ is a weak solution of $(1.1) \Leftrightarrow J'(u_0) = 0$.

1.2 Conditions on a C¹-function defined on a Real Banach Space 1.2 Conditions on a C¹-function defined on a Real Banach Space **1.2 Conditions on a** *C***¹ -Function Defined on a Real Banach Space** linear functional having the following expression,

Define a functional, J : H¹

0 (Q) −
| 2 (Q) −→ R as, R a

Given a real *Banach Space X* and a C¹-function, $I: X \to \mathbb{R}$, our primary objective in this section shall be to obtain certain conditions on I in order to ensure that, $\exists x_0 \in X$ satisfying, $I'(x_0) = 0$. eal *Banach Space X* and a C^1 -function, $I: X \to \mathbb{R}$,

 $G_{\rm eff}$ real $G_{\rm eff}$ and a real $G_{\rm eff}$ and $G_{\rm eff}$ in this in th en, for every $x_1, x_2 \in \mathbb{R}$ with, $x_1 < x_2$, if $\exists x_3 \in (x_1, x_2)$ satisfying, $I(x_3) > max\{I(x_1), I(x_2)\}\)$, it follows that, $\exists x_0 \in (x_1, x_2)$ that, $I'(x_0) = 0$. The answer is quite simple for the one-dimensional case. Consider the example when, $X = \mathbb{R}$. Correspondingly, choose any C^1 -function, $s_{\rm eff}$ shall be to obtain conditions on Γ in order to ensure that, Γ \sim $\frac{1}{2}$ I: $\mathbb{R} \to \mathbb{R}$. Then, for every $x_1, x_2 \in \mathbb{R}$ with, $x_1 < x_2$, if $\exists x_3 \in (x_1, x_2)$ satisfying, $I(x_3) > max\{I(x_1), I(x_2)\}\$, it follows that, $\exists x_0 \in (x_1, x_2)$ such that $I'(x_1) = 0$ \sim \sim μ that, $I'(x_0) = 0$.

Although, for higher dimensional cases, it's rather difficult. For example, if we consider, $I: \mathbb{R}^2 \to \mathbb{R}$ defined as, $I(x, y) := e^x - y^2$. We can Figure dimensional cases, it is failed dimension, I choosingly, I we consider, I : ∞ are defined as, $I(x, y)$. $C = y$ or any (x_1, y_1) and $(x_1, -y_1)$ in \mathbb{R}^2 with $x_1, y_1 > 0$, $I(x_1, \pm y_1) < 0$ and, $I(x, 0) = e^x >$ $\pm y_1$) where, deduce that, for any (x_1, y_1) and $(x_1, -y_1)$ in \mathbb{R}^2 with $x_1, y_1 > 0$, $I(x_1, \pm y_1) < 0$ and, $I(x, 0) = e^x > 0$. This implies that, the line, $y = 0$ separates the noints $(x + y)$ where \mathbf{r} , I⁷ $\pm y_1$) where, Although, for higher dimensional cases, it is rather different cases, it we consider \mathcal{L} the points $(x_1, \pm y_1)$ where,

$$
I(x,0) > max\{I(x_1,y_1), I(x_1,-y_1)\}, \quad \forall x \in \mathbb{R}
$$

tant observation that, $I'(x, y) = (e^x, -2y)$ helps us conclude that, \sharp any critical *point* for *I*. I : R2 → R2 → R2 → R2 → R2 → R2 → Y2. We can deduce that, for any (x1, y1) and But, an important observation that, $I'(x, y) = (e^x, -2y)$ helps us conclude that, $\neq \exists$ any critical *point* for *I*.

let A construction and the open.

$\overline{1}, \overline{G}$ with \overline{G} and \overline{G} and \overline{G} and \overline{G} and \overline{G} and \overline{G} and, \overline{G} and, \overline{G} and \overline **1.3 Palais-Smale Condition**

define a functional, J : H11 : H

Definition 1.3.1. (Palais-Smale Condition) Given a Banach Space X and, $I \in C^1(X;\mathbb{R})$, we define the functional I to satisfy the Pal- λ_n , $\lambda_{n,l}$ is $\lambda_{n,l}$ and $\lambda_{n,l}$, $\lambda_{n,l}$, $\lambda_{n,l}$ 2. Smale Condition, if every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which $I(x_n)$ is *bounded* and $I'(x_n) \to 0$ contains a convergent subsequence.

We can infer about the above problem as discussed in the section (1.2) in case for an arbitrary Banach Spaces using a famous result by *Ambrosetti* and *Rabinowitz*. *Ambrosetti* and *Rabinowitz*. in R2 with $\frac{1}{2}$ of $\frac{1}{2}$ and, I(xi) $\frac{1}{2}$ and, if $\frac{1}{2}$ implies that, the line, y $\frac{1}{2}$ and $\frac{1}{2}$

 ζ ψ $=$ **Theorem 1.3.1.** (Mountain-Pass Theorem) Given a Banach Space X and, $I \in C^1(X; \mathbb{R})$, as But, an important observation that, I′ tion. Furthermore, suppose, $\exists R > 0$ and, $e \in X$ satisfying, $||e|| > R$ and, $b = \inf_{x \in \partial B_R(0)} I(x) > \max\{I(0), I(e)\}$. Then, $\exists x_0 \in X$ such that, $I(x) > 0$ and. $I(x) > b$. **Theorem 1.3.1.** (Mountain-Pass Theorem) Given a Banach Space X and, $I \in C^1(X;\mathbb{R})$, assume that, I satisfies the Palais-Smale Condi- $I'(x_0) = 0$ *and*, $I(x_0) \ge b$. $I(x)$ > max $\{I(0), I(e)\}$. Then, $\exists x_0 \in X$ such that,

Remark 1.3.2. In other words, the theorem implies that, if a pair of points in the graph of I are indeed separated by a mountain $\partial B_R(0)$, then *I* has a *critical point*.

Remark 1.3.3. One of our aims in this article is to present aproof of the *Mountain-Pass Theorem* (1.3.1), as well as applying the statement of the same in order to look for non-negative weak solution $u \in H^1_0(\Omega)$ to the non-linear boundary value problem (1.1) over a
bounded domain $\Omega \subset \mathbb{R}^n$ f: $\mathbb{R} \to \mathbb{R}$ being continuous bounded domain $\Omega \subset \mathbb{R}^n$, $f : \mathbb{R} \to \mathbb{R}$ being continuous.

1.4 Differentiation in Banach Spaces

1.4 Differentiation in Banach Spaces
Suppose, we choose any two real Banach Spaces *X* and *Y* , and, *B*(*X*; *Y*) denotes the space of all bounded linear operators from *X* to *Y*. Let us further denote, $X^* := B(X; \mathbb{R})$. Moreover, let $A \subset X$ be *open*. $\mathbf{1}$.

Definition 1.4.1. Let, $x_0 \in A$. A function, $I: A \to Y$ is defined to be *Frechet Differentiable* (or, just *differentiable*) at x_0 , if \exists a bounded linear operator $D(x) \cdot Y \to Y$ in other words $D(x) \in B(Y, Y)$ satisfying linear operator, $DI(x_0)$: $X \to Y$, in other words, $DI(x_0) \in B(X; Y)$ satisfying, α

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 2

$$
\lim_{h \to 0} \frac{||I(x_0 + h) - I(x_0) - DI(x_0)h||_Y}{||h||_X} = 0
$$
\n(1.5)

Example 1.4.1. For any differentiable function, $f: \mathbb{R} \to \mathbb{R}$ at some $x_0 \in \mathbb{R}$, we have, $f'(x_0) \in \mathbb{R}$. Correspondingly, the Frechet Derivative, $Df(x_0)$ of f at the point x_0 has the following expression, $Df(x_0)(x) = f'(x_0)x$, $\forall x \in \mathbb{R}$. 0 \mathbf{R} = \mathbf{R} = \mathbf{R} and \mathbf{R} at \mathbf{R} and \mathbf{R} are the free \mathbf{R} defined as \mathbf{R} and \mathbf{R} The derivative of Indianapolis $\lim_{x_0} \sum_{k=1}^{\infty} x_k$, we have, $y(x_0) \in \mathbb{R}$. Solver pointingly, the Freeher By the point x_0 has the following expression, $Df(x_0)(x) = f'(x_0)$, $\forall x \in \mathbb{R}$. **Definition** 1. Pot any different able function, $J : \mathbb{R} \to \mathbb{R}$ at some $x_0 \in \mathbb{R}$, we have, $J(x_0) \in \mathbb{R}$. Correspondingly, the Fiechet Defined to be point x has the following expression $Df(x)(x) = f'(x)$ x $\forall x \in \math$ \mathbf{F} and \mathbf{F} and the Article at \mathbf{F} and \mathbf{F} is continuous. If, \mathbf{F} is continuous.

In general, in case when, $X = \mathbb{R}$, $A = (a, b)$ and, $I : A \to Y$ be *differentiable* at $x_0 \in (a, b)$, then we define, $I'(x_0) := DI(x_0)(1) \in Y$ via the canonical isomorphism, $i : B(\mathbb{R}; Y) \to Y$ given by, $i(T) := T(1)$. $\frac{d}{dx}$ at every point of A. Furthermore, I $\frac{d}{dx}$ is continuous. I $\frac{d}{dx}$, $\frac{d}{dx}$ is continuous. I $\frac{d}{dx}$, $\frac{d}{dx}$,

 $(x_0, y_0; \mathbf{f}(\mathbf{f}), \mathbf{f})$ is Y via the canonical isomorphism, i.e. \mathbf{f} , \mathbf{f} the comparation \mathbf{f} the field Derivative of I at x_0 defined as, $DI(x_0)$ satisfying (1.5), we write $I'(x_0)$ to be the derivative of I at x_0 . For arbitrary real Banach Spaces X and Y, if I: $A \to Y$ be Frechet Differentiable at some $x_0 \in A$, then, corresponding to the Frechet
Derivative of Lat x, defined as $D(x)$ satisfying (1.5), we write $I'(x)$ to be the deriv Definition 1.4.2. A function of a solid $\sum_{i=1}^n a_i$ is defined as $D(x_i)$ satisfying (1.5), we write $I'(x_i)$ to be the derivative of I at x_i . α is a set of A. Furthermore, I α if α If $(x_0, D_1(x_0))$ substitute that, (x_0, y_0) is so that definitive to If X and Y , if $I : A \rightarrow Y$ be Frechet Differentially

Definition 1.4.2. A function, $I: A \to Y$ is defined to be differentiable on A if it is differentiable at every point of A. Furthermore, $I \in C1(A;$ Y) if, I' : $A \rightarrow B(X; Y)$ is continuous. $E(X, \lambda)$ suppose, (X, λ , .) be a Hilbert Space and, I : X λ is λ and, I : X λ $D(x, T)$ is commodes.

Example 1.4.2. (*i*) Suppose, $(X, \langle ., . \rangle)$ be a *Hilbert Space* and, $I: X \to \mathbb{R}$ defined as, $I(x) = \langle x, x \rangle$. Then, $I \in C^1(X; \mathbb{R})$ such that, uppose. $(X, \langle ... \rangle)$ be a *Hilbert Space* and, $I: X \rightarrow \mathbb{R}$ d . (*i*) Suppose, $(X, \langle ., . \rangle)$ be a *Hilbert Space* and, $I : X \rightarrow \mathbb{R}$ defined as,

$$
I'(x)y = 2\langle x, y \rangle, \qquad x, y \in X.
$$

 (1) (2) . Define, $\frac{1}{2}$: H₀(2). The dist, (*ii*) Suppose, $f \in H^{-1}(\Omega)$. Define, $J : H^1_{(0)}(\Omega) \to \mathbb{R}$ as,

$$
j(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u.
$$

where that, $J \subseteq C$ (H₁⁰(Sz), S₃, F and Christian conclusion We can in fact conclude that, $J \in C^1(H^1_0(\Omega);\mathbb{R})$. Furthermore,

 $\mathcal{L}(\mathcal{$

$$
J'(u)\phi = \int_{\Omega} \nabla u \nabla \phi - \int_{\Omega} f \phi \qquad \forall u, \phi \in H_0^1(\Omega).
$$

We can comment on some important properties of the derivative as follows. The derivative as follows. The derivative as follows. on some important properties of the derivative as follows. We can comment on some important properties of the derivative as follows.

 B . If, I and J are differentiable at $x_0 \in A$ and $I(x_0)$ respectively, then, $J \circ I : A \to Z$ is differentiable at x_0 , and, 1. priori given X, Y and Z, and, A ⊂ X, B ⊂ Y being non-empty open sets, suppose, I: A → Y and J: B → Z be t
1. B J f J and J are differentiable at $x \in A$ and $I(x)$ respectively then $J \circ J : A \to Z$ is differentiable at x, a 1. (Chain Rule) A priori given X, Y and Z, and, $A \subset X$, $B \subset Y$ being non-empty open sets, suppose, $I : A \to Y$ and $J : B \to Z$ be functions satisfying, $I(A) \subseteq B$. If, I and J are differentiable at $x_0 \in A$ and $I(x_0)$ respectively, then, $J \circ I : A \to Z$ is differentiable at x_0 , and,

$$
(J \circ I)'(x_0) = J'(I(x_0)) \circ I'(x_0).
$$
 (1.6)

2. (Mean Value Theorem) A priori given a differentiable function, $I: A \to Y$ and, $x_0, x_1 \in A$, let us define,

$$
[x_0, x_1] = \{\lambda x_0 + (1 - \lambda)x_1 : 0 \le \lambda \le 1\}
$$

 $2.$ (Mean Value Theorem) A priori given a differentiable function, I α α β ent in A.. Then, ent in A.. Then, to be a line segment in A.. Then, to be a line segment in A.. Then, to be a *line segment* in A.. Then,

$$
||I(x_1) - I(x_0)|| \le \sup_{x \in [x_0, x_1]} ||I'(x)|| ||x_1 - x_0||. \tag{1.7}
$$

 $X, I : A \longrightarrow Y$ being differentiable at some $x \in A$, let us 3. (Taylor's Formula) A priori given a function, $I: A \to Y$ being differentiable at some $x \in A$, let us define,

|
|-
| R(h)|| R(h)

$$
R(h) := I(x+h) - I(x) + I'(x)(h)
$$

 $|f$ if ested to $|f(x)|$ $\mathbf{f}_{\alpha\beta}$ $|f|$ if $|f(x)|$ Then, by (1.5), the Remainder Term, $R(h)$ satisfies,

$$
\lim_{h \to 0} \frac{||R(h)||}{||h||} = 0, \quad \text{i.e., } R(h) = o(||h||). \tag{1.8}
$$

Theorem 1.4.3. Suppose, $\Omega \subset \mathbb{R}^n$ be bounded and open, and, $p > 1$. Moreover, let $g : \mathbb{R} \to \mathbb{R}$ be a C^1 -function satisfying, $T_{\rm Edd} = 1.1.311$ ηg ,

||R(h)||

||R(h)||

||R(h)||

lim

lim

lim

||R(h)||

lim

lim

||R(h)||

- 1. $|g(t)| \leq a + b|t|^p$, $\mathcal{Q}[t]$,
	- $b|t|^{p-1}$ 2. $|g'(t)| \leq a + b|t|^{p-1}$

for some constants a, b. Also,

lim

||R(h)||

$$
I(u) := \int_{\Omega} g(u(x)).
$$
\n(1.9)

Then, $I \in C^1(L^p(\Omega);\mathbb{R})$, and, for every $u \in L^p(\Omega)$,

$$
I'(u)\phi = \int_{\Omega} g'(u)\phi, \qquad \forall \phi \in L^{p}(\Omega).
$$

Corollary 1.4.4. Suppose, for any continuous function, $f: \mathbb{R} \to \mathbb{R}$ satisfying,

$$
|f(t)| \le a + b|t|^p
$$

 (s) de be the prime $f(x) \leq \frac{(n+2)}{(n-2)}$. Moreover, let, $F(t) = \int_{0}^{t} f(s)ds$ be the primitive of f. If, $I : H_0^1(\Omega) \to \mathbb{R}$ has the foll $f(x) \leq \frac{(n+2)}{(n-2)}$. Moreover, let, $F(t) = \int_{0}^{t} f(s)ds$ be the primitive of f. If, $I : H_0^1(\Omega) \to \mathbb{R}$ has the foll = 0, i.e., $R(h) = o(||h||)$. (1.8)

> 1. Moreover, let g : ℝ → ℝ be a C'-function satisfying,
 $\big) := \int_{\Omega} g(u(x))$. (1.9)
 $\Omega' (u) \phi, \quad \forall \phi \in L^p(\Omega)$.

ℝ satisfying,
 $(t)| \le a + b|t|^p$

as be the primitive of f. If, I : $H_0^1(\Omega) \to \$ $f(x) \leq \frac{(n+2)}{(n-2)}$. Moreover, let, $F(t) = \int_{0}^{t} f(s)ds$ be the primitive of f. If, $I : H_0^1(\Omega) \to \mathbb{R}$ has the foll f for every, $1 < p \leq \frac{(n+2)}{(n-2)}$. Moreover, let, $F(t) = \int_{0}^{t} f(s)ds$ be the primitive of f . If, $I : H_0^1(\Omega) \to \mathbb{R}$ has the foll $\boldsymbol{0}$ $f(s)ds$ be the primitive of f. If, $I: H_0^1(\Omega) \to \mathbb{R}$ $definition,$ for every, $1 < p \leq \frac{(n+2)}{(n-2)}$. Moreover, let, $F(t) = \int_{0}^{t} f(s)ds$ be the primitive of f. If, $I : H_0^1(\Omega) \to \mathbb{R}$ has the following for every, $1 < p \leq \frac{(n+2)}{(n-2)}$. Moreover, let, $F(t) = \int_{0}^{t} f(s)ds$ be the primitive of f. $f(\textbf{s})$ is be the primitive of f. If, $I : H_0^1(\Omega) -$

$$
I(u) := \int_{\Omega} F(u(x)) dx.
$$

Then, $I \in C^1(H^1({\Omega}); \mathbb{R})$, and,

$$
I'(u)\phi = \int_{\Omega} f(u)\phi \qquad \forall \phi \in H_0^1(\Omega). \tag{1.10}
$$

2 Critical Points $\sum_{i=1}^{n}$

2 Critical Points
In this section, we assume everywhere unless mentioned otherwise, that, X is a Banach Space, and $A \subset X$. $\mathcal{L} = \mathcal{L} = \mathcal$ x_0 , we assume everywhere unless mentioned otherwise, that, X is a *Banach Space*, and $A \subseteq X$.

 $\mathcal{S}_{\mathcal{D}}$ S_{S} is defined by S_{S} minimum) of *I* if, $\exists U(x_0)$ of x_0 in *X* satisfying, **Definition 2.0.1.** (Minima of a Function) Given a function, $I: A \to \mathbb{R}$, we define a point $x_0 \in A$ to be a local minimum (resp. strictly local \mathcal{S} is defined by \mathcal{S}

$$
I(x_0) \leq I(x) \qquad \forall \ x \in U(x_0) \cap A \tag{2.1}
$$

resp.,

$$
I(x_0) < I(x) \quad \forall \ x \in U(x_0) \cap A \setminus \{x_0\} \tag{2.2}
$$

On the other hand, x_0 is defined to be a global minimum of *I* if,

$$
I(x_0) \le I(x) \qquad \forall \ x \in A \tag{2.3}
$$

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 4 $T₁$ \sim $T₂$ $\$

 \overline{Y} $\overline{$

 \mathbf{C} and \mathbf{D} is \mathbf{D} in \math Therefore, we write, $I(x_0) = \min_{x \in A} I(x)$. $\overline{}$ to be a local maximum (resp. strictly local maximum) of $\overline{}$ if, $\overline{}$ if $\overline{}$

$$
I(x_0) \ge I(x) \qquad \forall \ x \in U(x_0) \cap A \tag{2.4}
$$

resp.,

$$
I(x_0) > I(x) \qquad \forall \ x \in U(x_0) \cap A \setminus \{x_0\} \tag{2.5}
$$

On the other hand, x_0 is defined to be a global maximum of *I* if,

$$
I(x_0) \ge I(x) \qquad \forall \ x \in A \tag{2.6}
$$

Therefore, we write, $I(x_0) = \max_{x \in A} I(x)$. $\overline{x \in A}$ is defined to be a global matrix of $\overline{x \in A}$ Therefore, we write, $I(x_0) = \max_{x \in A} I(x)$. $\mathcal{L}(\mathcal{L})$ suppose, A $\mathcal{L}(\mathcal{L})$

 $x \in A$ $x \in A$ $\in A$ $\in \mathbb{R}$ and $I: A =$ $R = \frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ is a $\frac{1}{2}$ in A. Then, and I is a $\frac{1}{2}$ is a $\frac{1}{2}$ in \frac **Remark 2.0.1.** Assume $A \subset X$ to be open, and $I : A \to \mathbb{R}$ be *diffeentiable* on A. Then, $E \in A$
 \Box **L.** Assume *A* ⊂ *X* to be open, and *I* : *A* → R be *diffeentiable* on *A*. Then,

Remark 2.0.1. Assume A ⊂ X to be open, and I : A → R be differentiable on A. Then, $\sum_{i=1}^{n}$ $I'(x_0)=0$ $I(x)$ \geq $I'(x_0) = 0$

 $D_{\rm eff}$ and $D_{\rm eff}$ and $D_{\rm eff}$ is a function) $G_{\rm eff}$: A μ : A μ : A μ : A μ is a point a point a point a point a point a point and μ

 $\mathbb{E} \left[\mathcal{L} \left(\mathcal{L} \right) \right]$ $\mathcal{L} \left(\mathcal{L} \right)$ $\mathcal{L} \left(\mathcal{L} \right)$ $\mathcal{L} \left(\mathcal{L} \right)$ $\mathcal{L} \left(\mathcal{L} \right)$

 $\mathcal{L}^{\mathcal{A}}$ to be a local maximum (resp. strictly local maximum) of $\mathcal{L}^{\mathcal{A}}$ if, $\mathcal{L}^{\mathcal{A}}$ is $\mathcal{L}^{\mathcal{A}}$

 $\overline{}$ to be a local maximum (resp. strictly local maximum) of $\overline{}$ if, $\overline{}$ if, $\overline{}$ in $\overline{}$ if, $\overline{}$ if, $\overline{}$ if, $\overline{}$ if, $\overline{}$ if, $\overline{}$ if $\overline{}$

 $\overline{}$ and $\overline{}$ if $\overline{}$ if, $\overline{}$ if, $\overline{}$ if, $\overline{}$ if, $\overline{}$ in $\overline{}$ if, $\overline{}$ if, $\overline{}$ if, $\overline{}$ in $\overline{}$ if, $\overline{}$ if, $\overline{}$ if

provided, x⁰ ∈ A is a local minima (or, a local maxima) of I. provided, $x_0 \in A$ is a local minima (or, a local maxima) of *I*. provided, $x_0 \in A$ is a local minima (or, a local maxima) of *I*. $A \subset \mathbb{R}^n$ is defined to be a critical point of $A \subset \mathbb{R}^n$ infered about the converse of the converse of

Definition 2.0.3. (Critical Point) Suppose, $A \subseteq X$ be open, and $I: A \to R$ be differentiable on A. A point $x_0 \in A$ is defined to be a critical point of I if, $I'(x_0) = 0$. Subsequently, $\overline{I}(x_0) \in \mathbb{R}$ is called the critical value. $f(x)$ on some subset A α are indeed critical points of I. However, the same cannot be same cannot **function** I on *A*. A point $x_0 \in A$ is defined to be a $x_0 \in A$ is defined to (λ_0) of Subsequently, $T(\lambda_0)$ $\subset \mathbb{R}$ is called the efficial value.

ical points of *I*. However, the same cannot be infered about the converse of this result. It is evident from the definition that, *local minima* and *local maxima* of a differentiable function *I* on some subset $A \subset X$ are indeed *crit*- $(x, y) = (2x, y)$, and, I°, $y = (2x, y)$ For the definition that, *local minima* and *local maxima* of a differentiable function *I* on some subset *A* Thus definition that, focal minima and focal maxima of a differentiative failed on $\frac{1}{2}$ of $\frac{1}{2}$ and $\frac{1}{2}$ are interesting to $\frac{1}{2}$.

For example, we take, $X = \mathbb{R}^2$, $I(x, y) = x^2-y^2$. Then, $I'(x, y) = (2x, -2y)$, and, $I'(0, 0) = 0$. Thus, $(0, 0)$ is a critical point of *I*, although, is neither a local minima nor a local maxima of *I*.
It follows from the fact that I_{α}

 μ follows from the fact that, It follows from the fact that,

$$
I(0,0) = 0, I(x,0) = x^2 > 0 \text{ for, } x \neq 0,
$$

$$
I(0, y) = -y^2 < 0 \text{ for, } y \neq 0
$$

Remark 2.0.2. Critical points satisfying the condition as above as termed as Saddle Points.

be a saddle point of I if, for every neighbourhood $U(x_0)$ of x_0 , $\exists x_1, x_2 \in U(x_0)$ satisfying, **Definition 2.0.4.** (Saddle Point) Suppose, $A \subset X$ be open, and $I : A \to \mathbb{R}$ be differentiable on A. We define a critical point $x_0 \in A$ of I to $U(x,y) = \sum_{i=1}^n \frac{1}{i!} \sum_{i=1}^n \frac{1}{i!} \sum_{j=1}^n \frac{1}{j!} \sum_{i=1}^n \frac{1}{j!} \sum_{j=1}^n \frac$

$$
I(x_1) < I(x_0) < I(x_2) \tag{2.7}
$$

Definition 2.0.5. (Convex Function) A function $I: X \to \mathbb{R}$ is defined to be convex, if $\forall x, y \in X$,

$$
I(tx + (1-t)y) \le tI(x) + (1-t)I(y) , \qquad \forall t \in (0,1)
$$

 $E_{\rm eff} = 2.0$, $E_{\rm eff} = 0.3$, $E_{\rm eff} = 0.3$ as, f(x) \sim $E_{\rm eff}$ as, f(x) \sim ex is \sim ex is \sim

 $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$: *Example* 2.0.3. Consider, $X = \mathbb{R}$. Then, the function, $f : \mathbb{R} \to \mathbb{R}$ defined as, $f(x) = e^x$ is convex everywhere on \mathbb{R} .

Remark 2.0.4. In case when, *I* is *convex* on *X*, the *critical points* of *I* are in fact global minima.

Proposition 2.0.5. *For a convex and differentiable function I* : $X \rightarrow \mathbb{R}$,

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 5

$$
I(x_0) \le I(x) \qquad \forall \ x \in X,
$$

i.e.,
$$
I(x_0) = \min_{x \in X} I(x).
$$

Proposition 2.0.6. Suppose, $I: X \to \mathbb{R}$ be a differentiable function satisfying the following condition, $\frac{1}{2}$

$$
I(y) \le I(x) + I'(y)(y - x) \qquad \forall \ x, y \in X
$$

 \forall ex. P_{cov} , we'll use the definition (2.0.5) of convexity. $\textit{Vex.}$ *Then, I is convex.*

Proof. To prove the result, we'll use the definition (2.0.5) of convexity. sider $z = \lambda x + (1 - \lambda)y$. \blacksquare Given that $I(y) \leq I(x) + I'(y)(y - x)$ for every $x, y \in X$.
For $\lambda \in [0, 1]$, consider $z = \lambda x + (1 - \lambda)y$. By the given condition: By the given condition: For $\lambda \in [0, 1]$, consider $z = \lambda x + (1 - \lambda)y$.

$$
I(z) \leq I(x) + I'(z)(z - x)
$$

= $I(x) + I'(z)(\lambda x + (1 - \lambda)y - x)$
 $\leq I(x) + \lambda I'(z)(x - y)$

Similarly,

$$
I(z) \leq I(y) + I'(z)(z - y)
$$

= $I(y) + I'(z)(\lambda x + (1 - \lambda)y - y)$
 $\leq I(y) + (1 - \lambda)I'(z)(x - y)$

 \mathbf{L} mequalities: Combining these inequalities: Combining these inequalities:

$$
I(z) \leq I(x) + \lambda I'(z)(x - y)
$$

\n
$$
\leq I(x) + \lambda I'(z)(x - y) + (1 - \lambda)I'(z)(x - y)
$$

\n
$$
= \lambda I(x) + (1 - \lambda)I(y) + \lambda(x - y)I'(z)
$$

 $\frac{J(x)}{x}$ is conv \overline{y} $\overline{I}(x)$ + (1 $\overline{I}(x)$) + (1 $\overline{I}(x)$) + $\overline{I}(x)$ + $\overline{I}(x)$ Since $\lambda \in [0, 1], I(z) \le \lambda I(x) + (1 - \lambda)I(y)$. Hence, $I(x)$ is convex.

the state the existence of critical points of a function, one needs to justify that the deand maxima respectively are well-defined. $\mathcal{L}_{\mathcal{L}}$ is the existence of contribution, one needs to $\mathcal{L}_{\mathcal{L}}$ function, one needs to $\mathcal{L}_{\mathcal{L}}$ Before we investigate the existence of *critical points* of a function, one needs to justify that, the definitions (2.0.1) and (2.0.2) of minima

 $\mathbf{B}(\mathbf{y}) = \mathbf{y}$ **Definition 2.0.6.** A priori, given X to be a *hausdorff* topological space, a function, $I: X \to \mathbb{R}$ is defined to be lower semi-continuous if \forall $c \in \mathbb{R}$, the set $\{x \in X \mid I(x) \le c\}$ is *closed*.

2.0.7. (i) Every continuous function is lower semi-continuous. $\mathbf{0.7}$. A priori, given $\mathbf{0.7}$ to be a hausdorff topological space, a function, I $\mathbf{0.7}$ $\mathbf{0.7}$ **Proposition 2.0.7.** (*i*) *Every continuous function is lower semi-continuous.*

s open, then, χ_A is lower semi-continuous. (*ii*) If $A \subset X$ *is open, then,* χ_A *is lower semi-continuous.*

Proof. (i) We intend to show that for any continuous function $f: X \to \mathbb{R}$, where X is a topological space, the set $\{x \in X : f(x) > a\} = A$ \mathbb{R}^n is \mathbb{R}^n in its implies that for any point \mathbb{R}^n is not chosen a holonic out of \mathbb{R}^n seem (say) is open for every $\alpha \in \mathbb{R}$. This implies that for any point x_0 in X, there exists a neighborhood U of x_0 such that $f(x) > \alpha$ for all x in U.

 $\frac{1}{\sqrt{1-\frac{1$ (ii) If \overline{a} is open, then, \overline{a} is lower semi-continuous. Let $\alpha \in \mathbb{R}$ be arbitrarily chosen. For any point $x_0 \in A$, since $f(x_0) > \alpha$, by the continuity of f , \exists a neighborhood U_{x0} of x_0 such that $f(x) >$ *α* ∀ *x* ∈ *U*_{*x*0}.

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 6 \mathbf{M} ek 2024 \mathbf{M}

2 CRITICAL PROPERTY AND RESIDENCE

Thus, for any $x_0 \in A$, there exists a neighborhood U_{x0} of x_0 contained in *A*, implying *A* is open. Thus, for any $x_0 \in A$, there exists a neighborhood U_{x0} of

Therefore, every continuous function $f: X \to \mathbb{R}$ is lower semi-continuous. \mathcal{L} To prove that the indicator function \mathcal{L} is lower semi-continuous, it lower semi-continuous, it is lower semi-continuous, it is lower semi-continuous, it is lower semi-continuous, it is lower semi-continuous $T_{\text{H}\text{C}\text{C}\text{O}\text{C}\text{C}}$, every commute semi-control

(ii) To prove that the indicator function χ_A of an open set $A \subset X$ is lower semi-continuous, it suffices to show that for any $x_0 \in X$, and for any α < 1, \exists a neighborhood U of x_0 such that $\chi_A(x)$ > α for all $x \in U$. $\{i\}$ To prove that the indicator function \mathcal{A} of an open set \mathcal{A} is lower semi-continuous, it is lower semi-continuous, it is lower semi-continuous, it is lower semi-continuous, it is lower semi-continuous, i

Formally, let χ_A be the indicator function of A, defined as: \mathcal{L} and \mathcal{L} be arbitrary, and let \mathcal{L} is open. Since A is open, \mathcal{L}

open.

 $\sigma_{\rm eff}$ and $\sigma_{\rm eff}$ and $\sigma_{\rm eff}$ such that f(x) $\sigma_{\rm eff}$

$$
\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}
$$

ry, and let α < 1 be given. Since A is open, ∃ a neighborhood U of x_0 contained in A. For any $x \in U$, $x \in A$, $x_0 < \alpha$ for an $x \in \mathbb{C}$. T Let $x_0 \in X$ be arbitrary, and let $\alpha < 1$ be given. Since A is open, \exists a neighborhood U of x_0 contained in A. For any $x \in U$, $x \in A$, so $\chi_A(x)$ = 1. Since α < 1, χ ₄ (*x*) > α for all $x \in U$.

Therefore, χ_A of an open set $A \subset X$ is indeed lower semi-continuous. we can consider $X = \mathbb{R}^n$ and, \mathbb{R}^n and, \mathbb{R}^n is independent and, \mathbb{R}^n is independent as in

Remark 2.0.8. A function which is *lower semi-continuous* may not be continuous. For example, we can consider $X = \mathbb{R}$, and, $A = (a, b)$ at the points *a* and *b* in ℝ. $\subset \mathbb{R}$ for any a, $b \in \mathbb{R}$ with $a \le b$. Then, χ_A is indeed lower semi-continuous on R (using above proposition), although it is not continuous

Theorem 2.0.9. A priori given X to be a compact and Hausdorff topological space. Furthermore, I : X \rightarrow R be lower semi-continuous. Then, I is bounded below, and, $\exists x_0 \in X$ such that,

$$
I(x_0) = \min_{x \in A} I(x).
$$

Theorem 2.0.9. Theorem 2.0.9. A priori given $H_n = \{x \in X \mid I(x) \geq -n\}$ is open $\forall n \in \mathbb{N}$. Moreover, $X = \bigcup_{i=1}^n A_i$ $\sum_{n=1}^{\infty}$ $\sum_{n=1}^{\infty}$ is bounded below. prover semi-continuous implies that, the set, $A_n = \{x \in X \mid I(x) \geq -n\}$ is open $\forall n \in \mathbb{N}$. Moreover, $X = \bigcup_{n=1}^{\infty} A_n$ ce, we must have, $X = \bigcup_{n=1}^{\infty} A_n$ for some $n_0 \in \mathbb{N}$. \mathbb{R}^n being lower semi-continuous implies that, the set, An $=$ $v \in \mathbb{N}$. Moreover, $X = \bigcup_{i=1}^{\infty} X_i$ $\bigcup_{n=1}^{\infty} A_n$, and, is *Proof.* I being lower semi-continuous implies that, the set open semi-continuous implies that, is compact. Hence, we must have, $X = \bigcup_{n=1}^{\infty} A_n$ for some $\bigcup_{n=1} A_n$ for some Proof. I being lower semi-continuous implies that, the set, $A_n = \{x \in X \mid I(x) > -n\}$ is open $\forall n \in \mathbb{N}$. Moreover, $X = \bigcup_{n=1}^{\infty} A_n$, and, is *compact. Hence, we must have,* $X = \bigcup_{n=0}^{\infty} A_n$ *for some* $n_0 \in \mathbb{N}$. we must have, $X = \bigcup_{n=0}^{n_0} A_n$ for some $n_0 \in \mathbb{N}$. As a consequence, I(x) $\Delta t = \bigcup_{n=1}^{\infty} t_n$ is a consequence that, X is bounded below. $\left| I(x) \right\rangle$ - n₃ is open \forall n $\in \mathbb{N}$. Moreover, $X = \bigcup_{n=1}^{\infty} A_n$, and, is $A \rightarrow A$

Example 1.1 $(X,Y) = n_0 Y X \in \Lambda$. Thus, it establishes that, A is bounded below.
Let us choose, inf $I = l$ (> - ∞). We claim that, such a "*l*" do exists. If not, then, we consider the collection $\{B_n\}_{n=1}^{\infty}$ where, B $P \times P$ is the set of S if \mathcal{L} is set of set of \mathcal{L} if \mathcal{L} is open-times, As a consequence, $I(x) > -n_0 \forall x \in X$. Thus, it establishes that, *X* is bounded below. $x \in X | I(x) > 1 + 1/n$ for every $n \in \mathbb{N}$ such that, $n=1$ $\sum_{n=1}^{\infty} n^{n}$ $\sum_{n=1}^{\infty} n^{n}$ As a consequence, $I(x) > -n_0 \forall x \in X$. Thus, it establishes that, X is bounded below. in $f = l$ (> – ∞). We claim that, such a "l" do exists. If not, then, we consider the collection ${B_n}_{n=1}^{\infty}$ where, $B_n := \frac{X}{l}$ As a consequence, $I(x)$ > $\frac{n_0}{x}$ $\frac{x}{x}$ is it establishes that, X is bounded below. \overline{X} = let \overline{X} = let us choose, infinite a \overline{X} do exists. If \overline{X} and \overline{X} consider the collection (B_n) ⊗ where B_n us choose, inf $I = l$ (> – ∞). We claim that, such a "l" do exists. If not, then, we consider the collection $\{B_n\}_{n=1}^{\infty}$ where, $B_n :=$ $x \in X \mid I(x) > 1 + 1/n$ for every $n \in \mathbb{N}$ such that,

$$
X=\bigcup_{n=1}^\infty B_n
$$

the compactness condition of X , $\exists n_1 \in \mathbb{N}$ satisfying, $X = \bigcup_{n=1}^{n_1} B_n$. In other words, the collection, $\{B_n\}_{n=1}^{n_1}$ inc $n=1$ ne co Bn. In other words, the complements condition $n=1$ is a finite subcover of $n=1$ A priori from the *compactness* condition of X , $\exists n_1 \in \mathbb{N}$ satisfying, $X = \bigcup_{n=1}^{n_1} B_n$. priori from the *compactness* condition of X , $\exists n_1 \in \mathbb{N}$ satisfying, $X = \bigcup_{i=1}^{n_1} B_i$. In other words, the coll finite subcover of *X* for some $n_1 \in \mathbb{N}$. $n_1 \in \mathbb{N}$. A priori from the *compactness* condition of X , $\exists n_1 \in \mathbb{N}$ satisfying, $X = \bigcup_{n=1}^{n_1} B_n$. In other words, the collection, $\{B_n\}_{n=1}^{n_1}$ indeed is a finite subcover of X for some $n \in \mathbb{N}$. $A = \frac{1}{2}$ n1 $\frac{1}{2}$ n1 $\frac{1}{2}$ n1 $\frac{1}{2}$ n1 $\frac{1}{2}$ n1 $\frac{1}{2}$ α is a finite subcover of α

 $S(1)$ is a finite subset of $\frac{1}{n}$ indices $I(x) > l + \frac{1}{n+1}$. $\forall x \in X$, a contradiction to the fact that, $l = \inf I$. $S > l + \frac{1}{n_1}$, $\forall x \in X$, a contradiction to the fact that, $l = \inf_{X} l$. $S \setminus l + \frac{1}{\sqrt{N}}$ $\forall x \in Y$ a contradiction to the fact that, $l = \inf l$ Subsequently, $I(x) > l + \frac{1}{n_1}$, $\forall x \in X$, a contradiction to the fact that, $l = \inf_{X} I$. $\frac{1}{1}$ sequently, $I(x) > l + \frac{1}{n_1}$, $\forall x \in X$, a contradiction to the fact that, $l = \inf_{X} I$.

inputation, we introduce the notion of sequentially lower semi-continuous. For the ease of computation, we introduce the notion of sequentially lower semi-continuous. For the ease of computation, we introduce the notion of sequentially lower semi-continuous.

DEFINITION 2.0.7. Given a *Hausdorff* topological space X, a function, $I : X \to \mathbb{R}$ is defined to be sequentially lower semi-contracted to X and Y . 0.7. Given a *Hausdorff* topological space X, a function, $I: X \rightarrow \mathbb{R}$ is defined to be sequentially lower semi-continuous for every sequence $\{x_n\}$ tending to x in X, **Definition 2.0.7.** Given a *Hausdorff* topological space X, a function, $I: X \to \mathbb{R}$ is defined to be sequenti be sequence $\{x_n\}$ tending to x in λ , D efinition 2.0.7. Given a Hausdorff topological space X , a function, I α is defined to α **Definition 2.0.7.** Given a *Hausdorff* topological space X, a function, $I: X \to \mathbb{R}$ is defined to be sequentially lower semi-continuous if,
for survey as well as the direct win X

$$
I(x) \le \lim_{n \to \infty} I(x_n) \tag{2.8}
$$

n→∞ Proposition 2.0.10. For a Hausdorff topological space X and a function I : X → R, the **n** 2.0.10. For a Hausdorff topological space X and a function $I: X \to \mathbb{R}$, the following holds tr **Proposition 2.0.10.** For a Hausdorff topological space X and a function $I: X \to \mathbb{R}$, the $\ddot{}$ P roposition 2.0 . For a μ and a function μ is μ and a function μ : X and a function μ **Proposition 2.0.10.** For a Hausdorff topological space X and a function $I: X \to \mathbb{R}$, the following holds true:

(2) Converse holds true only if X is a metric space.

Thus, for any x0 ∈ A, there exists a new york a new york a new york a new york a in A, implying A is a interest

(1) *I* is lower semi-continuous \Rightarrow *I* is sequentially lower semi-continuous.
(2) C

(2) *Converse holds true only if X is a metric space*.

Proof. (1) Given $I: X \to \mathbb{R}$ to be lower semi-continuous. Suppose, (x_n) be a sequence in X converging to x, and suppose $I(x_n) \to I(x)$. We wish to show that $I(x) \le \liminf_{n \to \infty} I(x_n)$. Since I is lower semi-continuous, for any $\epsilon > 0$, \exists a neighborhood U of x such that $I(y) > I$ (*x*) − ϵ ∀ *y* ∈ *U*.

Since $x_n \to x$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $x_n \in U$. I(y) > I(x) − ϵ ∀ y ∈ U. $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ \overline{X} + \overline{Y} + \overline{Y} + \overline{Y} $\frac{1}{n}$ $\frac{1}{n}$

Thus, for every $n \ge N$, $I(x_n) > I(x) - \epsilon$. $T_1, T_2, T_3, T_4, T_7, T_8, T_9$

Taking $n \rightarrow \infty$,

$$
\liminf_{n \to \infty} I(x_n) \ge I(x) - \epsilon
$$

Since ϵ was arbitrary, we conclude:

$$
I(x) \le \liminf_{n \to \infty} I(x_n)
$$

Therefore, I is sequentially lower-semi continuous, and the proof is thus complete.

(2) Assume $I: X \to \mathbb{R}$ be a sequentially lower semi-continuous function, and let $\alpha \in \mathbb{R}$ be arbitrary. It suffices to establish that, the set *A* := $\{x \in X : I(x) \ge \alpha\}$ is open in X. $x := \{x \in X : I(x) > \alpha\}$ is open in X.
Let x_0 be any point in the set, say, $\{x \in X : I(x) > \alpha\}$, i.e., $I(x_0) > \alpha$.

 \mathcal{L} is defined by \mathcal{L} indian Definition \mathcal{L} Subham De 11 IIT Delhi, India

Since *I* is sequentially lower semi-continuous, for any sequence (x_n) in *X* converging to x_0 , we have $I(x_0) \le \liminf_{n \to \infty} I(x_n)$. \mathcal{A} as a result, \mathcal{A} n \mathcal{A} n \mathcal{A} is a sequence ally lower semi-continuous, for any sequence (x_n) in X converging to x_0 ,

Since x_0 is in the set $\{x \in X : I(x) > \alpha\}$, we have $\liminf_{n \to \infty} I(x_n) > \alpha$.

As a result, $\exists N \in \mathbb{N}$ such that $\forall n \ge N$, $I(x_n) > \alpha$ [\because the limit inferior of a sequence is the greatest lower bound of the set of subsequential limits].

Therefore, for any sequence (x_n) in X converging to x_0 , \exists a neighborhood U of x_0 such that $I(x) > \alpha$ for all $x \in U$, which implies that $\{x \in U\}$ $\in X$: $I(x) > \alpha$ } is open in X.

Since α was arbitrary, this holds for all $\alpha \in \mathbb{R}$. Therefore, we can conclude that, *I* is lower semi-continuous.

Remark 2.0.11. A sequentially lower semi-continuous function on a non metrizable space may not be lower semi-continuous. Consider the example where, $X = \mathbb{R}$ is equipped with the cofinite topology.

Define the function $f: X \to \mathbb{R}$ as follows:

$$
f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}
$$

Consider any sequence (x_n) in X converging to x. Since every neighborhood of x in the cofinite topology contains all but finitely many points of X, x_n equals x for all sufficiently large n. Therefore, $\liminf_{n\to\infty} f(x_n) = f(x)$, which makes $f(x)$ sequentially lower semi-continuous.

Let's examine the point $x = 0$. The set $\{x \in X : f(x) > 0\}$ is the set of irrational numbers in X, which is not open in the cofinite topology because it contains infinitely many points. Hence, $f(x)$ is not lower semi-continuous.

 X , which is not open in the cofinite topology because it contains it contains it contains infinitely many points. Hence, \mathcal{A} **Proposition 2.0.12.** *Given a Hausdorff topological space X and, I : X* → ℝ, *suppose that, for every* $c \in \mathbb{R}$,

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 8

The set
$$
\{x \in X : I(x) \le c\}
$$
 is **compact**,
$$
(2.9)
$$

Then, I is bounded below, and, $\exists x_0 \in X$ *satisfying,* $I(x_0) = \inf_{x \in X} I(x)$.

Proof. To prove this statement, we'll use the fact that compactness of the level sets of *I* implies certain properties about *I*.

unboundedness of I. This contradicts the assumption that all such sets are compact. Hence, I must be bounded below. Suppose *I* is not bounded below. Then, for each $n \in \mathbb{N}$, $\exists x_n \in X$ such that $I(x_n) < -n$. However, this implies that the set $\{x \in X : I(x)$ ≤ − *n*} is nonempty and contains the sequence {*xn* }, but it cannot be compact as the sequence has no convergent subsequence due to the

Since I is bounded below, let $c = \inf_{x \in X} I(x)$. Then, for each $n \in \mathbb{N}$, $\exists x_n \in X$ such that $I(x_n) < c + 1$ *n*. Consider the sequence $\{x_n\}$. Since $\{x \in X : I(x) \le c + 1/n\}$ is compact, there exists a subsequence $\{x_{nk}\}\$ converging to some x_0 in X. By continuity of I, $I(x_{nk}) \to I(x_0)$. But since $I(x_{nk}) \le c + 1/n_k$, we have $I(x_0) \le c$. Conversely, since c is the greatest lower bound of *I*, we have $I(x_0) \ge c$. Therefore, $I(x_0) = c$, and x_0 is the desired point.

Remark 2.0.13. Consider, *X* = ℝ. Define a function, *I* : ℝ → ℝ as, *I* (*x*) = *e* [→]. We can observe that, I is indeed smooth and bounded below, but I does not achieve its infimum. Applying Theorem (2.0.9) and Proposition (2.0.10) in order to obtain infimum, a compactness condition either for the space or, for the function must be required.

In case for infinite dimensional Banach Space *X*, the compactness condition is not achieved under the norm topology. However a certain amount of compactness is achieved to ensure the attainment of the infimum can be obtained in weaker topology on *X*.

In the next section, we shall learn more about a weaker topology called *weak topology* on *X* as compared to the norm topology defined on it.

3. Weak Topology on Banach Spaces

3.1 Weak Convergence

Suppose we consider $(X, \|\cdot\|)$ to be a Banach Space with X^* as its *dual*. Moreover, τ be the metric topology on *X* induced by the norm $\|\cdot\|$, having the following definition, $d(x, y) = ||x - y||$, $\forall x, y \in X$.

We intend to define the weakest topology τ_w on *X* as follows : "*Every functional* $f \in X^*$ *is continuous on X with respect to the topology τw on X*". The topology *τw* thus formed is defined as the Weak Topology on *X*. As for convergence of any sequence under *weak topology*, we provide the following definition. \mathcal{L} force wing definition.

Definition 3.1.1. (Weak Convergence) A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to converge to $x \in X$ weakly, and is denoted by, $x_n \to x$, if,

$$
f(x_n) \longrightarrow f(x) \qquad \forall \ f \in X^*
$$

Remark 3.1.1. If a sequence $\{x_n\}_{n=1}^{\infty}$ converges to x with respect to the norm $||.||$ (i.e., $||x_n - x|| \to 0$ as $n \to \infty$), we assert that, x_n converges to x strongly in X, in other words, $x \to x$. to *x* strongly in *X*, in other words, $x_n \to x$.

 $\frac{1}{n}$ *Proposition* **3.1.2.** We can accumulate some important properties of weak convergence of sequences as follows:

 \rightarrow x is unique. 1. $x_n \rightarrow x$ *in* $X \rightarrow x$ *is unique.*

2. $x_n \to x$ in $X \to x_n \to x$. The converse is not true in general

3. $x_n \to x \Longrightarrow \{||x_n||\}_{n=1}^{\infty}$ *is bounded, and,* $||x|| \le \liminf_{n \to \infty} ||x_n||$.

This implies, $||.|| : X → ℝ$ *is weakly sequentially lower semi-continuous.*

4. Suppose, *X* be reflexive. Furthermore, $||x_n|| \le M \forall n \in \mathbb{N}$ for some, $M > 0$. Then, $\exists x_0 \in X$ and a subsequence, $\{x_n\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ n=1 in X such that, $x_n \to x_0$. *X* such that, $x_{n_k} \rightarrow x_0$.

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 9 This implies, $|X|$ is weakly separate semi-continuous. In this weakly semi-continuous.

5. Let, Y be another Banach Space and $T \in B(X, Y)$. Then,

• $x_n \rightarrow x \Rightarrow Tx_n \rightarrow T_x$. • If *T* is compact and, $x_n \to x \Rightarrow Tx_n \to T_x$ in *Y*. $\sum_{i=1}^{n} T_{x}$. $\frac{1}{\sqrt{N}}$ and $\frac{1}{\sqrt{N}}$ and $\frac{1}{\sqrt{N}}$ and, f(xn) is a sequence, $\frac{1}{N}$

6. If X be a Hilbert Space, Then, $x_n \to x$ and, $||x_n|| \to ||x|| \Rightarrow x_n \to x$.

Proof. 1. Suppose, $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that, $x_n \to x$ and, $x_n \to y$. Then, for every $f \in X$, $f(x_n) \to f(x)$ and, $f(x_n) \to f(x)$ (v). Since, $\{f(x_n)\}$ is a sequence of numbers, hence its limit is unique, i.e., $f(x) = f(y)$, i.e., for every $f \in X^*$, we have, $f(x - y) = 0$. \mathcal{U}_n and \mathcal{Y}_n be two sequences in A su \mathcal{C} is not true in general. The intervalse in \mathbf{V}

Therefore, using Corollary (4.3.4) (cf. [24, Pg 223]), we conclude, $x - y = 0$, and thus, the weak limit is indeed unique. $\mathcal{L} \left(\mathcal{L} \right)$ and all other terms are $\mathcal{L} \left(\mathcal{L} \right)$ when $f(x, t)$ (cf. $[2, t, 1]$ and $[2, 0]$, we concrete $x, x \in \mathcal{Y}$ collary $(4, 3, 4)$ (cf. [24, Pg. 2231), we conclude $x - y = 0$.

2. By definition, $x_n \to x$ means, $||x_n - x|| \to 0 \Rightarrow$ For every $f \in X^*$, \mathbb{L} bounded linear functional on \mathbb{L} for all n. Since u_{max} the n-th coordinate, $\frac{1}{\sqrt{2}}$ ν_{λ} along the n-th conductive vector $y \in A$,

$$
|f(x_n) - f(x)| = |f(x_n - x)| \le ||f|| \cdot ||x_n - x|| \longrightarrow 0.
$$

Implying that, $x_n \rightharpoonup x$. \mathcal{L} . inequality). Therefore, f(en) α for all n implies that f does not converge to 0 as n α 0 as n α Hence, xⁿ = eⁿ converges weakly to 0. Hence, xⁿ = eⁿ converges weakly to 0.

Converse is not true in general. For strong convergence, we need to show that ∥xⁿ − 0∥ = ∥en∥ → 0 as n → ∞. However, For strong convergence, we need to show that ∥xⁿ − 0∥ = ∥en∥ → 0 as n → ∞. However, $\frac{1}{2}$ strong convergence, we need to show that $\frac{1}{2}$ θ For strong convergence, we need to show that ∥xⁿ − 0∥ = ∥en∥ → 0 as n → ∞. However, μ for all n, so μ and μ and μ are not tend to 0.000 μ and the 0.000 μ and the 0.000 μ

Consider the sequence $\{x_n\}$, where, $x_n = e_n$, $\forall n \in \mathbb{N}$ in ℓ^2 , where e_n is the sequence whose *n*-th term is 1 and all other terms are 0. Let f be any bounded linear functional on ℓ^2 . Then $f(e_n) = 1$ for all n. Since en is the unit vector along the n-th coordinate, $|f(e_n)| = ||f|| ||e_n||$ For any bounded mean functional on v . Then $f(e_n) = 1$ for all *n*, since e_n is the unit vector along the n-un coordinate, $y(e_n) = ||f||$ (by the Cauchy-Schwarz inequality). Therefore, $f(e_n) = 1$ for all *n* implies that f d converges weakly to 0. For strong convergence, we need to show that $||x_n - 0|| = ||e_n|| \to 0$ as $n \to \infty$. However, $||e_n|| = 1$ for all n , so $||x - 0|| = 1$ for all n , which does not tend to 0. $||e_n|| = 1$ for all *n*, so $||x_n - 0|| = 1$ for all *n*, which does not tend to 0. α is a convergence, we need to show that $||x - \theta|| = ||e|| \rightarrow 0$ as $n \rightarrow \infty$. However, $\frac{3.6}{100}$ Schwarz inequality). Therefore, $f(e_n) = 1$ for all *n* implies that f does not converge to 0 as $n \to \infty$. Hence

Therefore, the sequence xn in ℓ^2 converges weakly to 0 but does not converge strongly. $\int \text{Re}(x) \, dx$ is $\int \text{Im}(x) \, dx$ is a constant depending on for converge strongly.

3. Given, $x_n \to x$. Thus, $f(x_n) \to f(x)$ $\forall f \in X \Rightarrow \{f(x_n)\}\$ is a convergent sequence of numbers, hence is bounded.

Let, $|f(x_n)| \le c_f \forall n \in \mathbb{N}$, where, c_f is a constant depending on f, but not on n. Using the canonical mapping, $C: X \to X^*$ (cf. (5) of Sec. 2.6, $y(x_n) \le c_f$ $v_n \in \mathbb{N}$, where χ^2 is a constant depending on j, but not on *n*. Osing the canonical is
4.6 [24, Pg. 240]), where χ^2 denotes the double dual of X, we can in fact define, $g_n \in \chi^2$ by, guide, we can in fact define, $g_n \in X^*$ to $f(x)$

$$
g_n(f) = f(x_n), \qquad f \in X^*
$$

Then,

$$
|g_n(f)| = |f(x_n)| \leq c_f \qquad n \in \mathbb{N}.
$$

Implying that, the sequence, $\{|g_n(f)|\}$ is bounded for every $f \in X^*$. Since, X^{*} is complete, by (2.10.4) (cf. [24, Pg. 120]), we can apply the Uniform Boundedness Theorem (cf. [24, Pg. 249]) to conclude that, $\{\|g_n\|\}$ Uniform Boundedness Theorem (cf. [24, Pg. 249]) to conclude that, $\{\Vert g_n \Vert\}$ is bounded.

 $(4.6.1)$ (cf. [24, Pg. 240]) helps us conclude that, $\{||xn||\}$ Now, $||g_n|| = ||x_n||$ by (4.6.1) (cf. [24, Pg. 240]) helps us conclude that, ${||x_n||}$ is bounded.

 $B_{\rm eff}(x) = \frac{1}{2} \left(\frac{1}{2} \frac{1}{\sqrt{2}} \right)$

lim inf

lim inf

In the set of the set of the statement is obviously true. Now, we assume $||x|| \neq 0$ By Theorem (4.3.3) $\equiv X^*$ such that, $\equiv X^*$ such that, As for the second part, if, $x = 0$, then, $||x|| = 0$, and the statement is obviously true. Now, we assume, $||x|| \neq 0$. By Theorem (4.3.3) (cf. [24, \mathbb{R}^{\times} such that, $\equiv X^*$ such that, Pg. 223]), ∃ some *f* ∈ *X** such that,

$$
||f|| = 1 , \quad f(x) = ||x||
$$

 $\lim_{n \to \infty}$ we have Since, $\{x_n\}$ converges weakly to x, and, f is indeed continuous, we have,

$$
\lim_{n \to \infty} f(x_n) = f(x) = ||x||.
$$

n→∞ ||xn|| ≥ limn→∞ f(xn) = ||x||.
|}

n→∞ ||xn|| ≥ limn→∞ f(xn) = ||x||.
|x|| ≥ limn→∞ f(xn) = ||x||.

 $||v|| \cdot ||v_n||$ $||v_n||$. Hence, $\text{But, } f(x_n) \leq |f(x_n)| \leq ||f|| \cdot ||x_n|| = ||x_n||.$ Hence,

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 10

$$
\liminf_{n \to \infty} ||x_n|| \ge \lim_{n \to \infty} f(x_n) = ||x||.
$$

in a reflexive Banach space, every bounded sequence has a weakly convergent subsequence.

4. To prove this statement, we can utilize the *Eberlein–Šmulian Theorem*, which states that in a reflexive Banach space, every bounded sequence has a weakly convergent subsequence. Since X is reflexive, every bounded sequence in X has a weakly convergent subsequence.

Let be a bounded sequence in X, i.e., $||x_n|| \le M$ for all $n \in \mathbb{N}$. By the Eberlein–Smulian Theorem, \exists a subsequence such that $x_{n_k} \to x_0$ in
Y Therefore $x \to x$ weakly where $x \in Y$ and is a subsequence of *X*. Therefore, $x_{n_k} \to x_0$ weakly, where $x_0 \in X$ and is a subsequence of .
The proof is thus complete. The proof is thus complete.

5. Given, $x_n \to x$ in X, thus, $f(x_n) \to f(x)$ for every $f \in X$. We intend to show that, \mathcal{S} is a compact operator from \mathcal{S} is and \mathcal{S} in \mathcal{S} in \mathcal{S} in \mathcal{S} is \mathcal{S} in \mathcal{S} in \mathcal{S} in \mathcal{S} is an \mathcal{S} in \mathcal{S} in \mathcal{S} is an \mathcal{S} in \mathcal{S} in \mathcal{S}

The proof is thus complete.

 $B_{\rm eff}(x) = \frac{1}{2} \int_{\rm eff}^{\rm eff} \left(\frac{1}{2} \frac{1}{\sqrt{2}} \right) \, d \mu$

$$
\varphi(T(x_n)) \longrightarrow \varphi(T(x)) \qquad \forall \ \varphi \in Y^*
$$

In other words,

$$
(\varphi \circ T)(x_n) \longrightarrow (\varphi \circ T)(x) \qquad \forall \varphi \in Y^*
$$

Although, $\varphi \circ T \in X^*$, therefore, our hypothesis guarantees our desired conclusion.
• Suppose T is a compact operator from X to Y, and $x_n \to x$ in X implies $Tx_n \to Tx$ in Y. By the definition of compact operators, every bou Suppose T is a compact operator nont X to T, and $x_n = x \ln X$ informs $Tx_n = Tx \ln T$. By the definition of compact operators, every bound-
ed sequence $\{x_n\}$ in X has a weakly convergent subsequence $\{x_n\}$ in X. Suppose, $x_n \$ Let $I x_{n_{k_j}} \to y$ in Y . $\text{Let } I \text{ and } I.$ Although, $\varphi \circ T \in X^*$, therefore, our hypothesis guarantees our desired conclusion. subsequence $\{Tx_{n_{k_j}}\}$ in *Y*. Let $Tx_{n_{k_j}} \to y$ in *Y*.

Now, we have $x_{n_{k_j}} \to x$ and $Tx_{n_{k_j}} \to y$. Since weak convergence implies boundedness, we have $\{x_{n_{k_j}}\}$ is bounded in X. By the first part of the statement, $Tx_{n_{k_j}}^T \rightarrow Tx$ in Y. But since $Tx_{n_{k_j}}$ converges to y in Y, by uniqueness of limits in Banach spaces, $Tx = y$. Hence, $Tx_n^T \rightarrow Tx$ in *Y* .

Therefore, if T is compact and $x_n \to x$ implies $Tx_n \to Tx$ in Y, and the statement holds true.

6. To prove this statement, let X be a Hilbert space, and suppose $x_n \to x$ weakly in X. Also, assume that $||x_n|| \to ||x||$. Since $x_n \to x$ weakly, for any $y \in X$, we have $\langle x_n, y \rangle \to \langle x, y \rangle$.

Now consider the sequence $y = x - x$ We have: $\sum_{n=1}^{\infty}$ $\sum_{n=1}^{\infty}$ is the state. $\frac{1}{n}$ is compact and xn $\frac{1}{n}$ in $\frac{1}{n}$ in $\frac{1}{n}$ in $\frac{1}{n}$, and the statement stat Now, consider the sequence $y_n = x_n - x$. We have:

$$
||y_n||^2 = \langle y_n, y_n \rangle = \langle x_n - x, x_n - x \rangle = ||x_n||^2 - 2\langle x_n, x \rangle + ||x||^2
$$

Given that $||x_n|| \to ||x||$, and $\langle x_n, y \rangle \to \langle x, y \rangle$ for any y, it follows that $\langle x_n, x \rangle \to \langle x, x \rangle$. $\langle v_1, v_2, v_3 \rangle$ and $\langle v_2, v_3 \rangle$ and $\langle v_3, v_4 \rangle$ and $\langle v_4, v_5 \rangle$

Thus, $||y_n||^2 = ||x_n||^2 - 2\langle x_n x \rangle + ||x||^2 \rightarrow ||x||^2 - 2\langle x, x \rangle + ||x||^2 = 0$ as $n \rightarrow \infty$. This implies that $||y_n|| \to 0$. But $||y_n|| = ||x_n - x||$, so we have $x_n \to x$ in X. I his implies that $||y_n|| \to 0$. But $||y_n|| = ||x_n - x||$, so we have $x_n \to x$ in X.
Therefore, if X is a Hilbert space, $x_n \to x$ weakly and $||x_n|| \to ||x||$, then $x_n \to x$ strongly in X.

Now, consider the sequence yⁿ = xⁿ − x. We have: Now, consider the sequence yⁿ = xⁿ − x. We have: **3.2 Existence of Minima** $\overline{\text{S}}$ of Minima

Theorem 3.2.1. Given a reflexive Banach Space $(X, ||.||)$, suppose, A be a weakly sequentially closed subset of X. Define, I: $A \rightarrow R$ satis*fying the following*: $\sum_{i=1}^{\infty}$ is contained on $\sum_{i=1}^{\infty}$ in A, i.e., i

Subham De 16 III De 16 II De 16 III De 16 II De 16
De 16 III De 16 II De 16 III De 16 III De 16 II De 16 II

 (I) (Compactness Condition) I is coercive on A, i.e., $\mathcal{L}(\mathcal{L})$ is condition) in the compactness coercive on $\mathcal{L}(\mathcal{L})$

 $I(u) \to \infty$ as $||u|| \to \infty$, $u \in A$.

 \overline{B} is the 16 indian Definition \overline{B} (II) *I* is weakly sequentially lower semi-continuous on A, i.e., if u_{n} , $u \in A$ with $u_{n} \to u$ in X, then,

 T is independent bounded by independent below, and, $\frac{1}{2}$ under that, $\frac{1}{2}$

$$
I(u) \le \liminf_{n \to \infty} I(u_n).
$$

ⁿ→∞ ^I(un).

I(u) ≤ lim info

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 11

nded below, and, ∃ u, ∈ A such that, *Then, I is indeed bounded below, and,* $\exists u_0 \in A$ *such that,*

$$
I(u_0) = \min_{u \in A} I(u).
$$

denote, $l := \inf \{I(u) | u \in A\}$. We intend on proving that, $l > -\infty$ and, $I(u_0) = l$ for some $u_0 \in A$. Suppose, $\{u_n\} \subset$ \mathcal{L} uo ∈ A satisfying, I(un) → l. ^I(u0) = min ^u∈^A $\text{fying, } I(u_n) \to l.$ Proof. Let $\lim_{u \to 0} f(I(u)) \cup I(u) = \lim_{u \to 0} f(I(u))$ intend on proving that, $I(u) = I$ for some $u \in A$. Sympage, $(u) \in I$ *Proof.* Let us denote, $l := \inf \{I(u) \mid u \in A\}$. We intend on proving that, $l > -\infty$ and, $I(u_0) = l$ for some $u_0 \in A$. Suppose, $\{u_n\} \subset A$ satisfying, $I(u_1) \to l$.

A priori using the fact that, *I* is coercive on *A*, $\exists M > 0$ such that, $||u_n|| \le M \forall n \in \mathbb{N}$. \mathbf{A} priori using that, I is considered on A, \mathbf{B} and \mathbf{B} and \mathbf{B} A priori using the fact that, *I* is coercive on *A*, $\exists M > 0$ such that, $||u_n|| \le M \forall n \in \mathbb{N}$. $\sum_{n=1}^{\infty}$ We are $\sum_{n=1}^{\infty}$

I(u) ≤ lim inf

 $T_{\rm eff}$ is independent bounded below, and, \bar{z} under that, \bar{z}

 \mathbb{Z} is a subsequence $\int_{\mathcal{U}} \mathbb{Q} \infty$ of $\int_{\mathcal{U}} \mathbb{Q} \infty$ with $u_n \to u$ in X for some $u \in Y$ Since, X is reflexive, \exists a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ with $u_{n_k} \to u_0$ in X for some $u_0 \in X$. 3 WEAK TOPOLOGY ON BANACH SPACES $u_n \to u_0$ in X for some $u_0 \in X$.

y weakly sequentially closed, we thus conclude that, $u_0 \in A$. Also, I is given to be weakly sequentially lower set
-1. \ldots 11which yields, tinuous on *A*, which yields, α weakly sequentially closed, we thus conclude that, $\mu \in A$. Also, I is given to be weakly sequentially lower set Moreover, A being weakly sequentially closed, we thus conclude that, $u_0 \in A$. Also, I is given to be weakly sequentially lower semi-contributions on A which vields $\alpha t, u \in A$ Also Lis given to be weakly sequentially lower set 3.3 Applications of \mathcal{A} being weakly sequentially closed, we thus conclude that, $u_0 \in A$. Alse

$$
l \leq I(u_0) \leq \liminf_{n \to \infty} I(u_{n_k}) = l = \inf_{u \in A} I(u).
$$

And the proof is complete. 3.3.1 Application in Linear PDEscription in Linear PDEscription in Linear PDEscription in Linear PDEscription i
2003 - PDEscription in Linear PDEscription in Linear PDEscription in Linear PDEscription in Linear PDEscriptio μ is complete.

3.3 Applications of the Existence of Minima W can in fact utilize T in order to find solutions to find solutions to find solutions to linear partial differential differential differential differential differential differential differential differential differen

3.3.1 Application in Linear PDEs equations in the following manner. equations in the following manner. Theorem 3.3.1. Suppose, ^Ω [⊂] ^Rⁿ be a bounded domain. For every ^f [∈] ^L2(Ω) (more generally,

We can in fact utilize Theorem $(3.2.1)$ in order to find solutions to linear partial differential equations in the following manner. The squadress in the forcem $\sum_{i=1}^{\infty}$ more generally. T_{max} 3.3.1. Suppose, T_{max} is a bounded domain. The above for every finite general linear general in the corresponding comments

Theorem 3.3.1. Suppose, $\Omega \subset \mathbb{R}^n$ be a bounded domain. For every $f \in L^2(\Omega)$ (more generally, $f \in H^1(\Omega)$), \exists a weak solution, $u_0 \in H_0^1$ $f(\Omega)$ to the following problem, $\frac{1}{0}$ **f** a Suppose, $\Omega \subset \mathbb{R}^n$ be a bounded domain. For every $f \in L^2(\Omega)$ (more generally, $f \in$ and in the following problem **f** a Suppose, $\Omega \subset \mathbb{R}^n$ be a bounded domain. For every $f \in L^2(\Omega)$ (more generally, $f \in$ a weaklem u = 0 on ∂Ω (3.1)

$$
\begin{cases}\n-\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega\n\end{cases}
$$
\n(3.1)

 $f(\mathbf{u}) = \mathbf{f}(\mathbf{u})$ $\mathbf{u} = \mathbf{f}(\mathbf{u})$

In other words, for every $\phi \in H^1_{\mathfrak{g}}(\Omega)$, In other words, for every ∞ H1
In other words, for every self-

$$
\int_{\Omega} \nabla u_0 \nabla \phi = \int_{\Omega} f \phi.
$$

3.3.2 Constrained Minimization **3.3.2 Constrained Minimization**

A priori given a Hilbert Space X over R and f,g ∈ C¹(H, R), let us define, $G := \{u \in H \mid g(u) = 0\}$. Moreover, let, $g'(u) \neq 0 \forall u \in H$ (This
3 is a manifold of so dimension 1) implies that, G is a *manifold* of co-dimension 1). A priori space X over R and i,g ∈ C·(H, R), let us define, G := {u ∈ H | g(u) = 0}. Moreover, let, g (u) \neq 0 v u ∈ G is a manifold of co-dimension 1). A priori given a Hilbert Space $\sum_{i=1}^{\infty}$ over $\sum_{i=1}^{\infty}$ and $\sum_{i=1}^{\infty}$ $\sum_{i=1}^{\infty}$ $\sum_{i=1}^{\infty}$ $\sum_{i=1}^{\infty}$ $\sum_{i=1}^{\infty}$ $\sum_{i=1}^{\infty}$ $\sum_{i=1}^{\infty}$ $\sum_{i=1}^{\infty}$ $\sum_{i=1}^{\infty}$ $\sum_{i=1}^{\infty}$ A priori given a *ritibert space A* over **R** and 1,g examples that, *G* is a *manifold* of co-dimension 1). A priori given a *Hilbert Space X* over R and f,g $\in C^1(H, \mathbb{R})$, let us define, $G := \{u \in H \mid g(u) = 0\}$. Moreover, let, $g'(u) \neq 0 \forall u \in H$ (This implies that. G is a *manifold* of co-dimension 1).

An important observation is, the *eradient* of ϱ , denoted a $\mathcal{A} = \mathcal{A} \cup \mathcal{A}$, let us define, G $\mathcal{A} = \mathcal{A} \cup \mathcal{A}$, let us define, G := {u $\mathcal{A} = \mathcal{A} \cup \mathcal{A}$ }, let us define, G := {u $\mathcal{A} = \mathcal{A} \cup \mathcal{A}$ }, let us define, $\mathcal{A} = \mathcal{A} \cup \mathcal{A}$ An important observation is, the *gradient* of g, denoted as $g'(u)$ is in fact normal to G. 0}. Moreover, let, g′ $\mathcal{C}=\{u\in \mathcal{C}^{\mathcal{C}}: u\in \mathcal{C$

Correspondingly, the *tangent space* T_u at $u \in G$ is defined as, Eft, the tangent space T_u at $u \in G$ is defined as, gly, the *tangent space* T_u at $u \in G$ is defined as, Correspondingly, the *tangent space* T_u at $u \in G$ is defined as,

$$
T_u := \{ v \in H \mid \langle g'(u), v \rangle = 0 \}.
$$

Definition 3.3.1. A point $u_0 \in G$ is defined to be a **critical point** of $(f|G)$, i.e., $(f|G)'(u_0) = 0$ if, **Definition 3.3.1.** A point $u_0 \in G$ is defined to be a **critical point** of $(f|G)$, i.e., $(f|G)'(u_0) = 0$ if,

 $u \in G$

$$
f'(u_0)(v) = 0 \qquad \forall \ v \in T_u.
$$

 T_{u0} being of co-dimension 1, we conclude that, $f'(u_0) = \mu g'(u_0)$, for some $\mu \in \mathbb{R}$. μ m and above μ in the above case is defined to be the **Lagrange Multiplier**. $\vec{v} = u\sigma'(u)$ for some $u \in \mathbb{R}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ T_{u0} being of co-dimension 1, we conclude that, $f'(u_0) = \mu g'(u_0)$, for some $\mu \in \mathbb{R}$. $\mathbf{e}^{\mathbf{e}}$ is defined to be the Lagrange Multiplier.

Proposition 3.3.2. If $\exists u_0 \in G$ and, $f(u_0) = min_{u \in G}$ {f(u)} $\in G$, then, we have, \cdot If $\exists u_0 \in G$ and, $f(u_0) = \min_{u \in G} \{f(u)\} \in G$, then, we have, $v = \frac{u \in G}{1 - \frac{u}{\sigma}}$ **Proposition 3.3.2.** *If* $\exists u_0 \in G$ *and*, $f(u_0) = \min_{u \in G} {f(u)} \in G$, then, we have,

$$
(f|_G)'(u_0) = 0.
$$

3.3.3 Application in Non-Linear PDEs 3.3.3 Application in Non-Linear PDEs 3.3.3 Application in Non-Linear PDEs tion in Non-Linear PDEs **solution** to the following non-linear PDEs A_{tot} be a N_{tot} . Domain. We choose, DDE_{tot}

3.3.3 Application in Non-Linear PDEs

3.3.3 Application in Non-Linear PDEs

Assume $\Omega \subset \mathbb{R}^n$ be a bounded domain. We choose, $\lambda \in \mathbb{R}$ and, $1 < p \leq \frac{(n+2)}{(n-2)}$. We intend to obtain a weak solution to the following non-linear Dirichlet boundary value problem, Assume $\Omega \subset \mathbb{R}^n$ be a bounded domain. We choose, $\lambda \in \mathbb{R}$ and, $1 < p \leq \frac{(n+2)}{(n-2)}$. We intend to obtain the property value and the set non-linear Dirichlet boundary value problem, Rⁿ be a bounded domain. We choose, λ ∈ R and, $1 < p \leq \frac{(n+2)}{(n-2)}$. We intend to obtain a weak solution to the f obtain a weak solution to the following non-linear Dirichlet boundary value problem, \mathbb{R}^n be a bounded domain. We choose, $\lambda \in \mathbb{R}$ and, $1 < p \leq \frac{(n+2)}{(n-2)}$. We intend to obtain a weak solution to the f \mathbb{R}^n be a bounded domain. We choose, λ ∈ R and, 1 < p ≤ $\frac{(n+2)}{(n-2)}$. We intend to obtain a weak solution to the f richlet boundary value problem, \mathcal{O} and the following non-linear Dirichlet boundary value problem, \mathcal{O} $\in \mathbb{R}$ and $1 < n < \frac{(n+2)}{2}$ We intend to obtain A s bounded domain. We choose, $\mathcal{L} = \{x \in \mathbb{R}^n : x \in \mathbb{R}^$ $\overline{}$ \mathbb{D}^n be a bounded domain We choose $\lambda \in \mathbb{D}$ and $1 \leq n \leq \binom{n+2}{1}$. We intend to obtain a weak solution −
p−1u = |u|
p−1u in 2u in 2u in 2u in 2u in 2u Assume $\Omega \subset \mathbb{R}^n$ be a bounded domain. We choose, $\lambda \in \mathbb{R}$ and, $1 < p \leq \frac{(n+2)}{(n-2)}$. We intend to obtain $\mathcal{O}(\log n)$ weak solution to the following non-linear Dirichlet boundary value problem, $\mathcal{O}(\log n)$ ose, $\lambda \in \mathbb{R}$ and, $1 < p < \frac{(n+2)}{2}$. We intend to obtain a weak solution to the following $\frac{1}{2}$ Assume $\Omega \subset \mathbb{R}^n$ be a bounded domain. We choose, $\lambda \in \mathbb{R}$ and, $1 < p \leq \frac{(n+2)}{(n-2)}$. We intend to obtain a weak solution to the following non-linear Dirichlet boundary value problem, Hon fincar Dirichnet ooundary variac problem,

non-linear Dirichlet boundary value problem,
\n
$$
\begin{cases}\n-\Delta u = |u|^{p-1}u + \lambda u & \text{in } \Omega \\
u \ge 0, u \ne 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega\n\end{cases}
$$
\n(3.2)

In other words, $\forall \phi \in H^1_{\mathfrak{0}}(\Omega)$, In other words, $\forall \phi \in H^1_0(\Omega)$,

$$
\int_{\Omega} \nabla u_0 \nabla \phi = \int_{\Omega} \int_{\Omega} |u|^{p-1} u_0 \phi + \lambda \int u_0 \phi. \tag{3.3}
$$

Taking, $\phi = u_0$ in (3.3), Taking, $\phi = u_0$ in (3.3),

$$
\int_{\Omega} |\nabla u_0|^2 = \int_{\Omega} = \int_{\Omega} |u|^{p+1} + \lambda \int u_0^2.
$$
\n(3.4)

Applying the **Sobolev Embedding Theorem**, we can in fact obtain the following embedding, *h* Moreover, $q = p + 1$ helps us conclude that, there indeed exist a *weak solution* Applying the Sobolev Embedding Theorem, we can in fact obtain the following embedding, **HERE CONSUMED SUBSUMPLE IS A SUMPLE TO SUMPLE THE UPPER CONSUMPLE THE UPPER CONSUMPLE TO A LARGE THE UPPER CONSUMPLE TO A LARGE THE P + 1 helps us conclude that, there indeed exist a** *weak solution* **to the equation (3.2)** Applying the Sobolev Embedding Theorem, we can in fact obtain the following embedding embedding embedding embedding, we can in fact obtain the following embedding, and the following embedding, and the following embedding, \mathbb{E}_{p+1} helps us conclude that, there indeed exist a *weak solution* to the equation (3.2) in $H_0^1(\Omega)$ for every $1 < p$ **Sobolev Embedding Theorem,** we can in fact obtain the following embedding, $H^1(\Omega) \to L^q(\Omega)$, where, $q \leq$ $= p + 1$ helps us conclude that, there indeed exist a *weak solution* to the equation (3.2) in $H^1_{\ 0}(\Omega)$ for every $1 < p$ in fact obtain the following embedding, $H^1_{\ 0}(\Omega) \to L^q(\Omega)$, where, $q \leq \frac{2n}{(n-2)}$. ed exist a *weak solution* to the equation (3.2) in $H^1_{\ 0}(\Omega)$ for every $1 < p \leq \frac{(n+2)}{(n-2)}$. Sobolev Embedding Theorem, we can in fact obtain the following embedding, $H_0^1(\Omega)$ in $H_2^1(\Omega)$ \cdot 1 helps us conclude that, there indeed exist a *weak solution* to the equation (3.2) in $H^1_{\ 0}(\Omega)$ for every $1 < p$ Sobolay Embedding Theorem, we can in fact obtain the following embedding H^1 (O), $I^q(0)$, where $q \leq$ **Sobolev Embedding 1 neorem**, we can in fact obtain the following embedding, $H_0^1(\Omega) - p + 1$ helps us conclude that, there indeed exist a *weak solution* to the equation (3.2) in H_0^1 $\frac{1}{\sqrt{2}}$ Applying the **Sobolev Embedding Theorem**, we can in fact obtain the following embedding, H_0^1 , H_0^2 Moreover, $q = p + 1$ helps us conclude that, there indeed exist a *weak solution* to $\mathbf d$ <mark>ng Theorem</mark>
clude that, tl $_{\rm re}$ \mathbf{c} n in fact obtain the following embedding, $H^1_0(\Omega)$
eed exist a *weak solution* to the equation (3.2) in *H* Applying the **Sobolev Embedding Theorem**, we can in fact obtain the following embedding, $H^1_{\ 0}(\Omega) \to L^q(\Omega)$, where, $q \leq \frac{2n}{(n-2)}$. Moreover, $q = p + 1$ helps us conclude that, there indeed exist a *weak solution* to the equation (3.2) in $H^1_{\ 0}(\Omega)$ for every $H_0^1(\Omega) \to L^q(\Omega)$, where, $q \leq \frac{2n}{(n-2)}$. 3.2) in $H_{0}^{1}(\Omega)$ for every $1 < p \leq \frac{(n+2)}{(n-2)}$. $\mathcal{A}_{0}^{(0)}(\Omega) \rightarrow L^{q}(\Omega)$, where, $q \leq \frac{2\pi}{(n-2)}$. $\begin{array}{lll} \n\hline\n\end{array}$ of $\hbox{S1}\leq P \leq (n-2)$. Applying the Sobolev Embedding Theorem, we can in fact obtain the following embedding, Moreover, $q = p + 1$ helps us conclude that, there indeed ex-0 (Ω) for every 1 = p ≤ (n+2)
2 (n+2) for every 1 = p ≤ (n+2) for every 1 = p

Suppose, $\phi \in H^1_{\phi}(\Omega)$ with $\phi \ge 0$ be the *Eigenfunction* corresponding to the first *Dirichlet Eigenvalue*, $\lambda_1(\Omega)$ of $-\Delta$ for the DIrichlet Suppose, $\varphi \in H_{0}(\Sigma)$ with $\varphi \ge 0$ be the *Eigenfunction* corresponding to the first *Dirichlet Eigenvalue*, $\kappa_1(\Sigma)$ or $-\Delta$ for the Dirichlet boundary condition as described in (3.2), having the following expressi $0 \leq \frac{1}{2}$ with $\frac{1}{2}$ or $\frac{1}{2}$ $\frac{1}{2}$ Suppose, $\phi \in H^1_{\mathfrak{g}}(\Omega)$ with $\phi \ge 0$ be the *Eigenfunction* corresponding to the first *Dirichlet Eigenvalue*, $\lambda_1(\Omega)$ of $-\Delta$ for the DIrichlet houndary condition as described in (3.2) having the following expre following expression, $\mathcal{L}(\mathcal{L})$ $\mathcal{L} = \frac{1}{\mathcal{L}}$ (n−2) . Suppose, $\varphi \in H_0(\Omega)$ with $\varphi \geq 0$ be the *Eigenfunction* corresponding to the following $\sum_{i=1}^n \sum_{i=1}^n \sum_{i$

$$
\lambda_1(\Omega):=\inf_{\phi\in H^1_0(\Omega)\setminus\{0\}}\left\{\frac{\int\limits_\Omega|\nabla\phi|^2}{\int\limits_\Omega\phi^2}\right\}
$$

Furthermore, from Furthermore, from (3.2), Furthermore, from (3.2), Furthermore, from (3.2), Furthermore, from (3.2),

$$
\begin{cases}\n-\Delta \phi = \lambda_1(\Omega)\phi & \text{in } \Omega \\
\phi = 0 & \text{on } \partial\Omega\n\end{cases}
$$
\n(3.5)

Proposition 3.3.3. For $1 < p \leq \frac{(n+2)}{(n-2)}$ and, $\lambda \geq \lambda_1(\Omega)$, the problem (3.2) admits no solution. on ∂

⇒ on ∂2.0) admits no solution. **Proposition 3.3.3.** For $1 < p \leq \frac{(n+2)}{(n-2)}$ and, $\lambda \geq \lambda_1(\Omega)$, the problem (3.2) admits no solution. $\frac{1}{\sqrt{2}}$ in $\frac{1}{\sqrt{2}}$ **Proposition 3.3.3.** *For* $1 < p \leq \frac{(n+2)}{(n-2)}$ and, $\lambda \geq \lambda_1(\Omega)$, the problem (3.2) admits no solution.

Sobolev Embedding Theorem ensures the existence of an embedding, $H^1(0)$, $\to L^{p+1}(\Omega)$, which is compact for every $p < \frac{2n}{(n-2)}$. We shall comment later on the case when $n - \frac{(n+2)}{n}$ $n > 3$ shall comment later on the case when, $p = \frac{(n+2)}{(n-2)}$, $n \ge 3$. or on the case when, $p = \frac{(n+2)}{(n-2)}$, $n \ge 3$. the case when, $p = \frac{(n+2)}{(n-2)}$, $n \ge 3$.

(n+2) Sobolev Embedding Theorem ensures the existence of an embedding, $H^1_0(\Omega)$, $\to L^{p+1}(\Omega)$, which is compact for every $p < \frac{2n}{(n-2)}$. We shall comment later on the case when $n = \frac{(n+2)}{n}$, $n > 3$

4. Assume, $1 < p < \frac{(n+2)}{(n-2)}$. For every $\lambda \in (-\infty, \lambda_1(\Omega))$, the Dirichlet boundary value problem (3.2) admits ther words $\exists u \in H^1(\Omega)$ satisfying $(n+2)$, ∞ = ∞ . **4.** Assume, $1 \leq p \leq \frac{(n-2)}{(n-2)}$. For every $\lambda \in (-\infty, \lambda_1(\Omega))$, the Dirichlet boundary value problem (3.2) admits ther words $\exists u \in H^1$. (O) satisfying $S_{\text{sum 1}}(n+2)$
Soume $1 \le n \le \frac{(n+2)}{(n-2)}$ For every $\lambda \in (-\infty, \lambda(\Omega))$ the Dirichlet boundary value problem (3.2) admits of $i.e., Assume, 1 \le p \le (n-2)$. For every $κ ⊂ (∞, κ_1(s2))$, the Dirichlet boundary value problem (5.2) damits ther words, $\exists u_0 \in H^1_0(\Omega)$ satisfying, **Theorem 3.3.4.** Assume, $1 < p < \frac{(n+2)}{(n-2)}$ and Figure . For every $\lambda \in (-\infty, \lambda_1(\Omega))$, the Dirichlet boundary value $S = \frac{1}{2}$ (2) $\frac{1}{2}$ (2) $\frac{1}{2}$ and $\frac{1}{2}$ **Theorem 3.3.4.** *Assume*, $1 \le p \le \frac{a-2}{n-2}$. *For every* $\lambda \in (-\infty, \lambda_1(\Omega))$, the Dirichlet boundary value problem (3.2) admits of a weak *solution. In other words*, $\exists u_0 \in H^1_0(\Omega)$ satisfying, **A.** Assume, $1 \le p \le \frac{(n-2)}{(n-2)}$. For every $\lambda \in (-\infty, \lambda_1(\Omega))$, the Dirichlet boundary value problem (3.2) admits a ther words $\exists u \in H^1$. (O) satisfying $\sum_{i=1}^{n} u_i \leq H_0$ (22) satisfying,

Subham De 19 III De

Subham De 19 III De

$$
\int_{\Omega} \nabla u_0 \nabla \phi = \int_{\Omega} |u_0|^{p-1} u_0 \phi + \lambda \int u_0 \phi.
$$
\n(3.6)

 $S_{\rm eff}$ Embedding Theorem ensures the embedding Theorem ensures the embedding, H11

 $\overline{}$ \overline

 $H^1_{0}(\Omega)$. $H_{0}^{\scriptscriptstyle L}(\Omega)$. *for every* $\phi \in H^1_0(\Omega)$. $\left(-1 \right)$:

Proof. Proof of the above result primarily hinges on two claims. First, we shall introduce some notations. $\frac{1}{\sqrt{2}}$ e above result primarily hinges on t *Proof.* Proof of the above result primarily hinges on two claims. First, we shall introduce some notations.
We define, $J: H^1(\Omega) \to \mathbb{R}$ as,

⁰ (Ω).

Ω

We define, $J: H^1_{\mathfrak{g}}(\Omega) \to \mathbb{R}$ as, $\frac{1}{2}$: $\frac{1}{2}$, $\frac{1}{2}$

Ω

$$
J(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} - \frac{\lambda}{2} \int_{\Omega} u^2.
$$

 $\forall g \in C^1(\mathbb{R})$ given by, $g(t) := |t|^{p-1}t + \lambda t$, and, $G(t) := \int_0^t g(s)ds$, we can in factinfer that, $\frac{1}{\sqrt{2}}$ $g \in C^1(\mathbb{R})$ given by, $g(t) := |t|^{p-1}t + \lambda t$, and, $G(t) := \int_0^t g(s)ds$, we can in factinfer that infer that, t, and, $G(t) := \int_{0}^{t} g(s) ds$, we can in fact
 $\frac{1}{s} \int_{0}^{t} g(s) ds + \int_{0}^{t} g(s) ds$ Moreover, for $g \in C^1(\mathbb{R})$ given by, $g(t) := |t|^{p-1}t + \lambda t$, and, $G(t) := \int_{a}^{t} g(s)ds$, we can in factinfer that, $\mathbf{0}$ $g(s)ds$, we can in facti

$$
G(u) = \frac{1}{p+1} \int_{\Omega} |u|^{p+1} + \frac{\lambda}{2} \int_{\Omega} u^2
$$

and, also, $G \in C^1(H^1(\Omega), \mathbb{R})$. Thus, $J \in C^1(H^1(\Omega), \mathbb{R})$ as well and alsio for every $u \in H^1(\Omega)$ and $\forall \phi \in H^1(\Omega)$,

$$
J'(u)\phi = \int_{\Omega} \nabla u \nabla \phi - \int_{\Omega} |u|^{p-1} u \phi - \lambda \int_{\Omega} u \phi.
$$
 (3.7)

sts that, if $u_0 \in H^1(0)$ with $J'(u_0) = 0$, then u_0 should satisfy (3.6). Hence, it only suffices to check for *critical poil* $\in H^1_{\mathfrak{g}}(\Omega)$. Therefore, for any $t \in \mathbb{R}$, $tu \in H^1_{\mathfrak{g}}(\Omega)$ as well, and, Choose $0 \neq u_1 \in H^1_0(\Omega)$. Therefore, for any $t \in \mathbb{R}$, $tu \in H^1_0(\Omega)$ as well, and, Which suggests that, if $u_0 \in H^1(\Omega)$ with $J'(u_0) = 0$, then u_0 should satisfy (3.6). Hence, it only suffices to check for *critical points* of *J*.

$$
J(tu_1) = \frac{t^2}{2} \int_{\Omega} |\nabla u_1|^2 - \frac{t^{p+1}}{p+1} \int_{\Omega} |u_1|^{p+1} - \frac{\lambda t^2}{2} \int_{\Omega} u_1^2.
$$

led below on $H^1_{\ 0}(\Omega)$, in other words, $J(tu_1) \to -\infty$ as $t \to \infty$, v $T_{0}^{(1)}(\Omega)$, in other words, fact that, J is unbounded below on $H_{0}^{1}(\Omega)$, in other words, J $(u_1) \to -\infty$ as $t \to \infty$, we conclude that, J does r
 $H_{0}^{1}(\Omega)$ $\overline{0}$ is unbounded below on H₁ $\overline{0}$ \vee $\overline{0}$ A priori from the fact that, J is unbounded below on $H_0^1(\Omega)$, in other words, $J(tu_1) \to -\infty$ as $t \to \infty$, we conclude that, J does not admit any minimum on $H^1_{\ 0}(\Omega)$.

 α al points of J, we denote, $\frac{1}{2}$ integral points of I , we denote To find critical points of *J*, we denote,

$$
A:=\left\{u\in H^1_0(\Omega)\mid\int\limits_{\Omega}|u|^{p+1}=1\right\}
$$

 $\lim_{n \to \infty} I \cdot A \to \infty$ as, and, $I: A \rightarrow \mathbb{R}$ as,

$$
I(u):=\frac{1}{2}\int\limits_{\Omega}|\nabla u|^{2}-\frac{\lambda}{2}\int\limits_{\Omega}u^{2}.
$$

As a result, using Theorem $(3.2.1)$, we state our first claim.

Lemma 3.3.5. For any $\lambda \in (-\infty, \lambda_1(\Omega))$, I is bounded on A. Further, $\exists u_{\lambda} \in A$ satisfying,

$$
I(u_\lambda) = \min_{u \in A} I(u).
$$

|u|

|u|

^p+1 [−] ¹

 $\frac{1}{1}$

On the other hand, considering, On the other hand, considering, On the other hand, considering,

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 14

$$
g(u):=\int\limits_{\Omega}|u|^{p+1}-1
$$

Definition (3.3.1), we can in fact, establish our second claim. $\frac{1}{2}$, $\frac{1}{2}$, and, subsequently, we define, $A := \{u \in H^1_0(\Omega \mid g(u) = 0)\}\.$ Hence, applying the concepts of *constrained minimization* as mentioned in T_{tot} is a weak solution of α solution of α implies

Lemma 3.3.6. *For every* $\lambda \in (-\infty, \lambda_1(\Omega))$, \exists *c_{* λ *}* > 0 *such that,*

$$
J'(c_{\lambda}u_{\lambda})=0.
$$

In in fact deduce that, $c_{\lambda} = (\mu(p+1))^{i(p-1)}$, for some $\mu \in \mathbb{R}_{+}$. \overline{c} In this case, we can in fact deduce that, $c_{\lambda} = (\mu(p+1))^{1/(p-1)}$, for some $\mu \in \mathbb{R}_{+}$.

Therefore, $\tilde{u}_{\lambda} = c_{\lambda} u_{\lambda}$ is a weak solution of (3.2). Furthermore, $u_{\lambda} \ge 0$, $u_{\lambda} \ne 0$ implying, $\tilde{u}_{\lambda} \ge 0$, and the proof is complete.

Lemma 3.3.6. For every 2.3.6. For every 2.3.6. For every 2.3.6. For every 2.3.6. For every 2.0 such that, ⊒ ca

Remark 3.3.7. For $p = \frac{(n+2)}{(n-2)}$, the embedding, $H^1(0) \to L^{p+1}(\Omega)$ is not compact. Under this scenario, the corresponding Dirichlet $T_1P = (n-2)$, the emocloding, T_0 (s2), T_1P (s2) is not compact. Onder this sechario, the corresponding is
roblem (3.2) is termed as the problem with lack of compactness, or, the critical exponent problem. As a result boundary value problem (3.2) is termed as the problem with lack of compactness, or, the critical exponent problem. As a result, the set
A as defined above during our second claim need not be weakly sequentially closed, and $\mathbf{1}$ boundary value problem (3.2) is termed as the problem with lack of compactness, or, the critical exponent problem. As a result, the set of weak solution in this case depends stricly on the choice of λ and the geometry of Ω This defined dove during our second claim need not be weaking sequentially crosed, and that our desired rest the existence of weak solution in this case depends stricly on the choice of λ and the geometry of Ω

(not given a locally linechite function $f: \mathbb{D} \to \mathbb{D}$ with $\Omega \subset \mathbb{D}^n$ being a bounded don \mathbf{s} . (Tonoguev s fuentity) A priori given a tocarry upschitz function, f . $\mathbf{x} \to \mathbf{x}$ with, sz $\subseteq \mathbf{x}$ being a bounded abo **Theorem 3.3.8. (Pohozaev's Identity)** A priori given a locally lipschitz function, $f: \mathbb{R} \to \mathbb{R}$ with, $\Omega \subset \mathbb{R}^n$ being a bounded domain with smooth boundary. Moreover, let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies, on the choice of \mathcal{L} and the geometry of \mathcal{L} $\frac{2}{3}$ We are substituted.

$$
\begin{cases}\n-\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega\n\end{cases}
$$
\n(3.8)

t
 $\frac{t}{\int f(x)dx}$ Suppose also that $y(x)$ is the unit outward normal at $x \in \partial \Omega$ and $\frac{\partial u}{\partial x} = \nabla u$ y Under these assi Theorem 3.3. (Pohos are significantly a priori given a local lipschitz function, $R = R + R + R$ $\geq R + R + R$ $\geq R + R + R + R + R + R$ $V(X) = \int_{0}^{x} f(s) ds$. Suppose also that, $v(x)$ is the unit outward normal at $x \in \partial \Omega$ and, $\frac{\partial}{\partial \nu}$ = vu.v. Under these assumed by *we have the following identity, Define*, $F(t) := \int_{0}^{t} f(s)ds$. Suppose also that, $v(x)$ is the unit outward normal at $x \in \partial \Omega$ and, $\frac{\partial u}{\partial \nu} = \nabla u.v$. Under these assumptions, ose also that, $v(x)$ is the unit outward normal at $x \in \partial \Omega$ and, $\frac{\partial u}{\partial \nu}$ = $\nabla u.v.$ Under these assu. μ assumptions, we have the following identity, we have the following identity, $\begin{array}{cc} t & \ & \text{if} \end{array}$ ose also that, $v(x)$ is the unit outward normal at $x \in \partial \Omega$ and, $\frac{\partial u}{\partial \nu}$ = $\nabla u.v.$ Under these assu. ∂u \boldsymbol{t} ose also that, v(x) is the unit outward normal at $x \in \partial\Omega$ and, $\frac{\partial u}{\partial \nu}$ = $\nabla u.v.$ Under these assu. $\overrightarrow{0}$

allowing identity, we have the following identity, we have the following identity, we have the following identity, $\overrightarrow{0}$ fine, $F(t) := \int_{0}^{t} f(s) ds$. Suppose also that, $v(x)$ is the unit outward normal at $x \in \partial \Omega$ and, $\frac{\partial u}{\partial x}$ $\stackrel{0}{\longrightarrow}$ Define, $F(t) := \int_{0}^{t}$ $\int_{0}^{x} f(s)ds$. Suppose also that, $v(x)$ is the unit outward normal at $x \in \partial \Omega$ and, $\frac{\partial a}{\partial \nu}$ $\frac{\partial u}{\partial \nu}$ = $\nabla u.v.$ Under these assumptions, Define, F(t) := $\delta_{\mathcal{F}}(f) := \int_{0}^{t} f(s)ds$. Suppose also that, $v(x)$ is the unit outward normal at $x \in \partial \Omega$ and, $\frac{\partial u}{\partial x}$ $\frac{0}{\omega}$ = 0

$$
2n\int_{\Omega} F(u) - (n-2) \int_{\Omega} f(u)u = \int_{\partial\Omega} x \cdot \nu \left(\frac{\partial u}{\partial \nu}\right)^2
$$
\nant derivation from the above identity, we might infer the following.

As an important derivation from the above identity, we might infer the following.

 S_{Coulon} 2.3.0 Assuming $O - D(D)$, the following Divisible type blow. C_1 . Assuming, Ω = $D(R)$, the following Dirichlet problem, **Corollary 3.3.9.** Assuming, $\Omega = B(R)$, the following Dirichlet problem,

$$
\begin{cases}\n-\Delta u = u^{\frac{(n+2)}{(n-2)}} + \lambda u & \text{in } B(R) \\
u \ge 0, u \ne 0 & \text{in } B(R) \\
u = 0 & \text{on } \partial B(R)\n\end{cases}
$$
\n(3.10)

2 2

doesn't admit any solution in $C^2(\Omega) \cap C^1(\overline{\Omega})$ for every $\lambda \in (-\infty, 0]$.

 \ddot{C} **Theorem 3.3.10.** $\exists \lambda_1(\Omega) > 0$ and, $\phi_0 \in H_0^1(\Omega)$, $\phi_0 \ge 0$, $\phi_0 \not\equiv 0$ satisfying, γ ing, γ ing, T_1 3.3.10. ∃ $\{Q\}$, Q_1 , I_2 $\{Q\}$, $\{Q_1\}$, Q_2 $\{Q_2\}$, Q_3 $\mathbf{10} \leftrightarrow \mathbf{10}$ **Theorem 3.3.10.** $\exists \lambda_1(\Omega) > 0$ and, $\phi_0 \in H_0^1(\Omega)$, $\phi_0 \ge 0$, $\phi_0 \not\equiv 0$ satisfying,

 $\overline{}$

Ξ

$$
\lambda_1(\Omega) = \int\limits_{\Omega} |\nabla \phi_0|^2
$$

ϕ⁰ ≥ 0, ϕ⁰ ̸≡ 0 in Ω

ϕ⁰ ≥ 0, ϕ⁰ ̸≡ 0 in Ω

ϕ⁰ ≥ 0, ϕ⁰ ̸≡ 0 in Ω

and,

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 15 24

^λ1(Ω)

$$
\begin{cases}\n-\Delta \phi_0 = \lambda_1(\Omega) \phi_0 & \text{in } \Omega \\
\phi_0 \ge 0, \ \phi_0 \ne 0 & \text{in } \Omega \\
\phi_0 = 0 & \text{on } \partial\Omega\n\end{cases}
$$
\n(3.11)

Furthermore, $\forall \phi \in H_0^1(\Omega)$, we must have,

and,

and,

$$
\lambda_1(\Omega) \int\limits_\Omega \phi^2 \leq \int\limits_\Omega |\nabla \phi|^2
$$

Proof. We define, Proof. We define,

$$
S; = \left\{ \phi \in H_0^1(\Omega) \mid \int_{\Omega} \phi^2 = 1 \right\}
$$

Hence, Hence,

and,

$$
\lambda_1(\Omega)=\inf_{\phi\in S}\left\{\int\limits_\Omega|\nabla\phi|^2\right\}
$$

Applying a result by *Poincare*, $\lambda_1(\Omega) > 0$. We choose a *minimizing sequence* $\phi_n \in S$, such that, u_z *ing sequence* $\varphi_n \in S$ Applying a result by *Poincare*, $\lambda_1(\Omega) > 0$. We choose a *minimizing sequence* $\phi_n \in S$, such that, $0 > 0$ α choose a *minimizin*

$$
\int_{\Omega} |\nabla \phi_n|^2 \longrightarrow \lambda_1(\Omega).
$$

 $\oint_{\mathcal{A}} \phi_n$ is bounded in $H^1_{\mathfrak{O}}(\Omega)$. Thus, \exists a sunsequence, $\{\phi_n\}_{n=1}^{\infty}$ of $\{\phi_n\}_{n=1}^{\infty}$ with $\phi_n \to \phi_0$ in $H^1_{\mathfrak{O}}(\Omega)$. From Rellich's Theoren ϕ deed compact. Hence, $\phi_{n_k} \to \phi_0$ in $L^2(\Omega)$, and, ϕ is bounded in $H^1(\Omega)$. Thus, \exists a sunsequence, $\{\phi_n\}_{n=0}^{\infty}$ of $\{\phi\}_{n=0}^{\infty}$ with $\phi_n \to \phi_n$ in $H^1(\Omega)$. From Rellich's Theorem deed compact. Hence, $\phi_{n_k} \rightarrow$ $\frac{M}{\sqrt{2\pi}}$ of $\frac{1}{\sqrt{2\pi}}$ of $\frac{1}{\sqrt{2\pi}}$ of $\frac{1}{\sqrt{2\pi}}$ of $\frac{1}{\sqrt{2\pi}}$ of $\frac{1}{\sqrt{2\pi}}$ (O). From Bill 12. The set φ_n is bounded in $H_{0}^{\sigma}(\Sigma)$. Thus, \exists a sunsequence, $\{\varphi_n\}_{n=1}^{\sigma}$ or $\{\varphi_n\}_{n=1}^{\sigma}$ with $\varphi_n \to \varphi_0$ in $H_{0}^{\sigma}(\Sigma)$. From Reflicti s Theorem ideed compact. Hence, $\varphi_n \to \varphi_n$ in $L^2(\Omega)$, and, $\frac{1}{\lambda}$ $\frac{1}{\lambda}$ in $\frac{1}{\lambda}$ ϕ is bounded in H¹ (Ω). Thus, ∃ a sunsequence, { ϕ_n , }∞ of { ϕ }∞, with $\phi_n \to \phi_n$ in H¹ (Ω). From Rellich's Theoren compact. Hence, $\phi_{n_k} \to \phi_0$ in $L^2(\Omega)$, and, Implying that, ϕ is bounded in H¹ (Q). Thus, ∃ a sunsequence, $\oint \phi_n \downarrow^{\infty}$ of $\oint \phi \downarrow^{\infty}$ with $\phi_n \rightarrow \phi$ in E $\lim_{\rho \to \infty} \lim_{\rho \to \infty} \lim_{n \to \infty} \frac{\log P_n}{\rho}$. From $\phi_n \to \phi_0$ in $L^2(\Omega)$, and, Indeed compact. Hence, $\phi_{n_k} \to \phi_0$ in $L^2(\Omega)$, and, Implying that, ϕ is bounded in H1(O). Thus, ∃ a sunsequence, (ϕ) ∞ of (ϕ) ∞, with $\phi \to \phi$ in H1(O). From 1 the pying that, φ_n is bounded in H₀(s2). Thus, \exists a sunsequence, $\varphi_{n,k}$ _i, or $\varphi_{n,k}$ _i, with $\varphi_{n,k}$ \rightarrow φ_0 in H₀(s2). From R₁, \rightarrow L²(Ω) is indeed compact. Hence, $\varphi_{n,k}$ \rightarrow φ_0 Implying that, ϕ_n is bounded in $H^1_{\,0}(\Omega)$. Thus, \exists a sunsequence, $\{\phi_n\}_{n=1}^{\infty}$ of $\{\phi_n\}_{n=1}^{\infty}$ with $\phi_n \to \phi_0$ in $H^1_{\,0}(\Omega)$. From \rightarrow $L^{2}(s)$ is indeed co Implying that, ϕ is bounded in $H^1(\Omega)$. Thus, \exists a sunsequence, $\{\phi_n\}_{n=1}^{\infty}$ of $\{\phi\}_{n=1}^{\infty}$ with $\phi_n \to \phi_n$ in $H^1(\Omega)$. From \rightarrow $L^2(\Omega)$ is indeed comp Implying that, ϕ_n is bounded in $H^1_{\ 0}(\Omega)$. Thus, \exists a sunsequence, $\{\phi_n\}_{n=1}^{\infty}$ of $\{\phi_n\}_{n=1}^{\infty}$ with $\phi_n \to \phi_0$ in $H^1_{\ 0}(\Omega)$. From Rellich's Theorem, $H^1_{\ 0}(\Omega)$ μ_{μ} , μ_{μ} is bounded in *L*₀(i.e.). Thus, \exists a subsequence, $(\psi_{\mu_k}\psi_{\mu_k})$ or $(\psi_n)_{n=1}$
 \rightarrow *L*²(Ω) is indeed compact. Hence, $\phi_{n_k} \rightarrow \phi_0$ in $L^2(\Omega)$, and,

$$
\lim_{k \to \infty} \int_{\Omega} \phi_{n_k}^2 = \int_{\Omega} \phi_0^2.
$$

and, $\int_{\Omega} \phi_0^2 = 1 \Rightarrow \phi_0 \in S$ Now, $\phi_{n_k} \in S$ and, $\int_{\Omega} \phi_0^2 = 1 \Rightarrow \phi_0 \in S$. Now, $\phi_{n_k} \in S$ and, $\int_{\Omega} \phi_0^2 = 1 \Rightarrow \phi_0 \in S$.

 $||.||_{H_0^1(\Omega)}$ being weakly. $\mathcal{L}_{H^1(\Omega)}$ being weakly sequentially lower semi-continuous implies, $H_0^1(\Omega)$ being weakly sequentially lower semi-continuous implies, $||.||_{H^1(O)}$ being weakly. \mathbf{U} F_0 (i.e.) rn Furthermore, $||.||_{H_0^1(\Omega)}$ being weakly sequentially lower semi-continuous implies, $\text{Fermor}, \frac{|\cdot|}{H_0^1(\Omega)}$ being weakly sequentially lower semi-continuous implies, Furthermore, $||.||_{H_0^1(\Omega)}$ being weakly sequentially lower semi-continuous implies,

$$
\lambda_1(\Omega) \leq \int_{\Omega} |\nabla \phi_0|^2 \leq \liminf_{k \to \infty} \int_{\Omega} |\nabla \phi_{n_k}|^2 = \lambda_1(\Omega).
$$

 $\Omega = \frac{J}{\Omega} + \nabla \varphi_0$. $\int \frac{|\nabla u|^2}{|u|^2}$ mg $\varphi_0 \leq 0$, $f(u) = \frac{9}{\Omega}$ |vu| and, $g(u) = \frac{9}{\Omega}$ $\alpha = 1$, where, $u \in S$, we obta $\frac{1}{\Omega} \frac{|\nabla \varphi_0|}{|\nabla \varphi_0|}$.
 $\frac{1}{\Omega} \frac{|\nabla u|^2}{|\nabla u|^2}$ and $\sigma(u) = \int u^2 = 1$, where $u \in S$ we obta Therefore, $\lambda_1(\Omega) = \int_{\Omega} |\nabla \phi_0|^2$. Ω Ω $\theta_0 \ge 0$, $f(u) = \frac{\int}{\Omega} |\nabla u|^2$ and, $g(u) = \frac{\int}{\Omega} u^2 - 1$, where, $u \in S$, we obta $\log \phi_0 \ge 0$, $f(u) = \frac{\int}{\Omega} |\nabla u|^2$ and, $g(u) = \int_{\Omega} u^2 - 1$, where, $u \in S$, we obta Moreover, assuming $\phi_0 \ge 0$, $f(u) = \int_{\Omega} |\nabla u|^2$ and, $g(u) = \int_{\Omega}$ suming $\phi_0 \ge 0$, $f(u) = \int_{\Omega} |\nabla u|^2$ and, $g(u) = \int_{\Omega} u^2 - 1$, where, $u \in S$, we obtain, \int_{Ω} | $\nabla u|^2$ and $\frac{32}{4}$ Therefore, $\lambda_1(\Omega) = \int_{\Omega} |\nabla \phi_0|^2$. Moreover, assuming $\phi_0 \ge 0$, $f(u) = \frac{\int}{\Omega} |\nabla u|^2$ and, $g(u) = \frac{\int}{\Omega} u^2 - 1$, where Moreover, assuming $\phi_0 \ge 0$, $f(u) = \int_{\Omega} |\nabla u|^2$ and, $g(u) = \int_{\Omega} u^2 - 1$, where, $u \in S$, we obtain, $= \int_{\Omega} |\nabla \phi_0|$ Moreover, assuming $\phi_0 \ge 0$, $f(u) = \int_0^u |\nabla u|^2$ and, $g(u) = \int u^2 - 1$, where, $u \in S$, we obtain, $\Delta u \in \mathcal{S}$, we obtain Therefore, $\lambda_1(\Omega) = \int_{\Omega} |\nabla \phi_0|^2$. | Moreover, assuming $\phi_0 \ge 0$, $f(u) = \frac{J}{\Omega} |\nabla u|^2$ and, $g(u) = \int_u^{\Omega} u^2 - 1$, where, $u \in S$, we obtain,

$$
g'(u)u = 2 \int_{\Omega} u^2 = 2 \neq 0.
$$

∇ϕ0∇ϕ = 2µ

∇ϕ0∇ϕ = 2µ

∇ϕ0∇ϕ = 2µ

Ω

Ω

Ξ

Ω

ϕ0ϕ

∇ϕ0∇ϕ = 2µ

ϕ0ϕ

ϕ0ϕ

Ω

Ξ

ΩÚ.

the concepts discussed in the section of Constrained Minimization, we get, μ amea μ μ m μ ıti μ , w A priori from the concepts discussed in the section of *Constrained Minimization*, we get. d in the section of C_0 $\mathbf c$ section of $\mathbf c$ is A priori from the concepts discussed in the section of *Constrained Minimization*, we get,

∇ϕ0∇ϕ = 2µ

Ω

Ω

 \mathbf{u}_1 ,

 $\mathfrak{a},$

 $n,$

$$
\int\limits_{\Omega}\nabla \phi_0\nabla \phi=2\mu\int\limits_{\Omega}\phi_0\phi
$$

A priori from the concepts discussed in the section of Constrained Minimization, we get,

Ω

ngrange Multiplier. Setting $\phi = \phi_0 \in S$, we obtain, $\lambda_1(\Omega) = 2\mu$. μ being the *Lagrange Multiplier*. Setting $\phi = \phi_0 \in S$, we obtain, $\lambda_1(\Omega) = 2\mu$.

 \mathbf{b} Applying (3.11) , $\Lambda = \frac{1}{2}$ (2.11) A ppig $\lim_{x \to 0}$ (3.11),

$$
\int_{\Omega} \nabla \phi_0 \nabla \phi = \lambda_1(\Omega) \int_{\Omega} \phi_0 \phi , \qquad \forall \phi \in H_0^1(\Omega).
$$

lude that, $\lambda_1(\Omega)$ and ϕ_0 indeed solve (3.11). We thus conclude that, $\lambda_1(\Omega)$ and ϕ_0 indeed solve (3.11). We thus conclude that, $\lambda_1(\Omega)$ and ϕ_0 indeed solve (3.11).

l. bsei s by definition of \overline{R} $F(\Omega)$, $u \not\equiv 0$, defining, $v = \frac{u}{\sqrt{1-u^2}}$, it can be obser $\sum_{i=1}^{n}$ $1/2$ that, $v \in S$ as well, on of $\lambda_1(\Omega)$, $F(\Omega)$, $u \not\equiv 0$, defining, $v = \frac{u}{\left(\int u^2\right)^{1/2}}$, it can be obser For any, $u \in H_0^1(\Omega)$, $u \not\equiv 0$, defining, $v = \frac{u}{\left(\int u^2\right)^{1/2}}$, it can be observed that, $v \in S$ as well, \mathcal{S} by definition of \mathcal{S} so by definition of $\lambda_1(\Omega)$, $\binom{u}{2}$ For any, $u \in H_0^1(\Omega)$, $u \not\equiv 0$, defining, $v = \frac{u}{\left(\int u^2\right)^{1/2}}$, it can be observed that, $v \in S$ as well, so by definition of $\frac{1}{2}$ $\int\limits_{\Omega} u^2$ $\sqrt{1/2}$, it can be observed that, $v \in S$ as well, ^λ1(Ω) ^u² [≤] n ing, $v =$ $\frac{u}{\left(\frac{u}{\rho u^2}\right)}$ Ω $\sqrt{2}$, it can be obser For any, $u \in H_0^1(\Omega)$, $u \neq 0$, defining, $v = \frac{u}{\left(\int_{\Omega} u\right)^{1/2}}$, it can be observed that, $\left(\begin{smallmatrix} 1 & a \\ \Omega & \end{smallmatrix}\right)$

Ω

Ω

$$
\lambda_1(\Omega) \leq \int\limits_\Omega |\nabla v|^2 = \frac{\int\limits_\Omega |\nabla u|^2}{\int\limits_\Omega u^2}.
$$

i.e., 1.6 .,

$$
\lambda_1(\Omega) \int_{\Omega} u^2 \le \int_{\Omega} |\nabla u|^2.
$$

le proof is complete. sequence of \mathbf{I} over X with certain conditions, which enables us to conditions, which enables us to conclude that, which enables us to conclude that, which enables us to conclude that, which enables us to conclude th And, hence the proof is complete. $\frac{1}{2}$ as $\frac{1}{2}$ or $\frac{1}{2}$ or $\frac{1}{2}$ with conditions, which enables us to conclude that, which expected s_{max} served the μ of μ over μ over μ over μ

4 Ekeland Variational Principle and its Applications
4.4 March 2010 March 2010 **4.1 Variational Principle.**

Assume, $T \in C(X, \mathbb{N})$ to be bounded below. Exclaine variational Principle-yields a sequence of *minimizers* of *P* over *A* with eer ditions, which enables us to conclude that, in various situations we can derive the inf *hare variational in interpic*
Assume, *I* ∈ *C*¹(*X*, ℝ) to be bounded below. Ekeland Variational Principle yields a sequaence of *minimizers* of *I* over *X* with certain conbeneficial for problems of minimax type. ditions, which enables us to conclude that, in various situations we can derive the infimum of I over X . Especially, this principle is quite α deficited to a problem semi-continuous function α cype.

Theorem 4.1.1. (Ekeland Variational Principle-Strong Form) Given a complete metric space (X, d) , and a lower semi-continuous As the contract of the bounded below, for every $\epsilon > 0$, $\lambda > 0$ and, $x_0 \in X$ satisfying, function $I \in C^1(X, \mathbb{R})$ which is bounded below, for every $\epsilon > 0$, $\lambda > 0$ and, $x_0 \in X$ satisfying,

$$
I(x_0) \leq \inf_X I + \epsilon,
$$

in various situations we can derive the infimum of \mathbb{R}^2 over \mathbb{R}^2 over \mathbb{R}^2 \tilde{z} and following conditions hold true: $\exists \bar{x} \in X$ such that, the following conditions hold true:

- (1) $I(\bar{x}) + (\frac{\epsilon}{\lambda}) d(\bar{x}, x_0) \leq I(x_0)$. λ) λ λ λ λ λ λ Ĵ
- $\mathcal{L}(\mathcal{N})$ $s = \frac{1}{\sqrt{2}}$ (2) $d(x_0, \bar{x}) \leq \lambda$.
- $\forall (x) + (\frac{\epsilon}{\lambda}) d(x, \bar{x}), \quad \forall x \in X, x \neq \bar{x}.$ $(2) + (6)$ d(x =) (3) $I(\bar{x}) < I(x) + \left(\frac{\epsilon}{\lambda}\right)d(x,\bar{x}), \quad \forall x \in X, x \neq \bar{x}.$

te. Moreover, for $x \in$ $\mathbb{E} \left[\left(\begin{array}{cc} (V, l), & 1 \end{array} \right) \right]$ *Proof.* Suppose, $d_1(x, y) := \frac{\epsilon}{\lambda} d(x, y)$. Then, (X, d_1) is complete. Moreover, for $x \in X$, we define,

$$
G(x) := \{ y \in X \mid I(y) + d_1(x, y) \le I(x) \}.
$$

The above definition yields the following:
(1) $x \in G(x)$ and, $G(x)$ is in fact closed The above definition yields the following:
(1) $x \in G(x)$ and, $G(x)$ is in fact closed. (2) *G*(*y*) ⊂ *G*(*x*) if, *y* ∈ *G*(*x*). (3) For $y \in G(x)$, we have, $d_1(x, y) \le I(x) - v(x)$, where, $v(x) := \inf_{z \in G(x)} I(z)$. $\frac{1}{2}$ γ ing: x) if, $y \in G(x)$.
(x), we have, $d_1(x, y) \le I(x) - v(x)$, where, $v(x) := \inf_{z \in G(x)} I(z)$. (x), we have, $d_1(x, y) \le I(x) - v(x)$, where, $v(x) := \lim_{z \in G(x)} I(z)$. $L(x, y) = \ln(2x) + 2$ is different and $L(x, y) = \ln(2x) + 2$. The above definition yields the following: $f(x)$ if, $y \in G(x)$. (2) $G(y) \subset G(x)$ if, $y \in G(x)$.

(3) For $y \in G(x)$, we have, $d_1(x, y) \le I(x) - v(x)$, where, $v(x) := \inf_{z \in G(x)} I(z)$.

from the triangle inequality. Subsequently, we can deduce (3) using definitions of G(x) and v(x).

A priori from the fact that, $I(.) + d_1(x,.)$ being lower semi-continuous, and by definition, we can assert that, $G(x)$ is indeed closed and, $x \in G(x)$. \mathcal{L} can assert that, G(x) is indeed closed and, x ∈ G(x). Let that, $I(.) + a_1(x,.)$ being *lower semi-continuous*, and by definition, we can assert that, $O(x)$ is indeed cite the fact that $I(x) + d(x)$ being lower semi-continuous, and by definition, we can assert that $G(x)$ is indeed clo fact that, $I(.) + d_1(x,.)$ being *lower semi-continuous*, and by definition, we can assert that, $G(x)$ A priori from the fact that, $I(.) + d_1(x,.)$ being *lower semi-continuous*, and by definition, we can asse $x \in G(x)$ \sum_{x} with, \sum_{x} wit $\chi \in G(x).$

 $\mathcal{L} = \langle \omega \rangle$

 \mathcal{A} priori from the fact that, I(.) \mathcal{A} and by definition, we define the finition, we define that, we define the finition, we define the finition, we define the finite of \mathcal{A}

Let, $z \in G(y) \Rightarrow I(z) + d_1(z, y) \le I(y) \Rightarrow I(y) + d_1(y, x) \le I(x)$. Hence, (2) follows from the *triangle inequality*. Subsequently, we can deduce (3) using definitions of $G(x)$ and $y(x)$. Starting from x, our objective is to construct a Let, $z \in G(y) \Rightarrow I(z) + d_1(z, y) \le I(y) \Rightarrow I(y) + d_1(y, x) \le I(x)$. Hence, (2) follows from the *triangle inequality*. Subseque (3) using definitions of $G(x)$ and $v(x)$. Starting from x_0 , our objective is to construct a sequence, $\{x_n$ (3) using definitions of $G(x)$ and $v(x)$. Starting from x_0 , our objective is to construct a sequence, $\{x_n\} \subset$

$$
x_{n+1} \in G(x_n) \ni I(x_{n+1}) \le v(x_n) + \frac{1}{2^n} \qquad \text{for } n \ge 0.
$$

Since, $v(x_0) = \inf \bigcup_{n=0}^{n} I(x_n)$, thus, $\exists x_1$ with,

 $\inf_{x \in G(x_0)} I(x)$, thus, $\exists x_1$ with, $=$ 0, and (2), and (2) Since, $v(x_0) = \inf_{x \in G(x_0)} I(x)$, thus, $\exists x_1$ with,

$$
x_1 \in G(x_0) \ni I(x_1) \le v(x_0) + \frac{1}{2}.
$$

ring $v(x_1)$, we obtain x_2 and so on. Using the fact that, $x_{n+1} \in G(x_n)$ for every $n \ge 0$, and (2), 2

Fing v(x), we obtain x, and so on Using the fact that $x \in G(x)$ for ever Now, considering $v(x_1)$, we obtain x_2 and so on. Using the fact that, $x_{n+1} \in G(x_n)$ for every $n \ge 0$, and (2), Now, considering $v(x_1)$, we obtain x_2 and so on. Using the fact that, $x_{n+1} \in G(x_n)$ for every $n \ge 0$, and (2),

$$
G(x_n) \supset G(x_{n+1}) \qquad \forall \ n \ge 0.
$$

n ≥ 0, and (2), and (ϵ and, We can derive that, σ Intersection Theorem, we get $N_{\rm eff}$ being a decreasing sequence of closed sequence of closed sequence of closed sets with diameter tending to 0

 α assert that, G(x) is independent and, α

$$
diam G(x_{n+1}) = \sup \{d_1(x, y) \mid x, y \in G(x_{n+1})\} \longrightarrow 0 \quad \text{as } n \to \infty.
$$

heing a *decreasing* s being a *decreasing sequence* of closed sets with diameter tending to 0, by Cantor's Intersection Theorem, we g
 $\frac{1}{2}$ $rac{100}{20}$ being a *decreasing sequence* of closed sets with diameter tending to 0, by C Now, $\{G(x_n)\}_{n\geq 0}$ being a *decreasing sequence* of closed sets with diameter tending to 0, by Cantor's Intersection Theorem, we get,

$$
\bigcap_{n=0}^{\infty} G(x_n) = \{\bar{x}\} \qquad \text{for some } \bar{x} \in X.
$$

 x is the required plut satisfying an the conditions as \overline{x} is the required pint satisfying all the conditions as described in the statement, \overline{x} , \overline{x} is the required pint satisfying all the conditions as described in the statement of the Theorem. We claim that, \bar{x} is the required pint satisfying all the conditions as described in the statement of the Theorem. of the Theorem.

 (x) $(x_0, \bar{x}) \leq I(x_0)$. Furthermore, $\bar{x} \in G(x_0) \implies I(\bar{x}) + d_1(x_0, \bar{x}) \leq I(x_0)$. Furthermore,

$$
d_1(\bar{x},x_0) \leq I(x_0) - v(x_0) \leq \epsilon.
$$

Important to observe that, $G(\overline{x}) = {\overline{x}}$. Then, for every $x \in X$, $x \neq \overline{x}$, we must have, $x \notin G(\overline{x})$. Consequently, $\sum_{i=1}^{n}$. Then, for every x ∈ $\sum_{i=1}^{n}$. Then, $\sum_{i=1}^{n}$

$$
I(x) + d_1(x, \bar{x}) > I(\bar{x}).
$$

 ϵ proof a priori using definition $d_1 = \frac{\epsilon}{2}d$ λ^{α} Example to observe that, G(x) $d_1 = \frac{\epsilon}{\lambda}d$. \triangle That completes the proof, a priori using definition, $d_1 = \frac{\epsilon}{\lambda}d$.

For the case, $\lambda = 1$, we can have a weaker version as follows.

2. (Ekeland Variational Principle-Weak Form) Assuming (X, d) to be a complete metric space, and $I: X \to \mathbb{R}$ is the and bounded below. For every chosen $\epsilon > 0$, $\exists x \in K$ satisfying, Γ important to observe that, G(x Γ) $=$ Γ $=$ Γ $=$ Γ $=$ Γ $=$ Γ $=$ Γ That completes the proof, a proof, a priori using definition, distribution, distr **Theorem 4.1.2. (Ekeland Variational Principle-Weak Form)** Assuming (X, d) to be a complete metric space, and $I: X \to \mathbb{R}$ be lower **FINDERET 4.1.2.** (*Exercity Variational Tracepte-weak Torm*) Assaming $\langle X, u \rangle$ is semi-continuous and bounded below. For every chosen $\epsilon > 0$, $\exists x_{\epsilon} \in X$ satisfying,

$$
(I) \ I(x_{\epsilon}) \le \inf_{x \in X} I(x) + \epsilon.
$$

$$
(II) \ I(x_{\epsilon}) < I(x) + \epsilon d(x, x_{\epsilon}) \qquad \forall \ x_{\epsilon} \in X, \ x_{\epsilon} \ne x.
$$

As a corollary, one can deduce the following. $\mathcal{F}_{\mathcal{A}}=\mathcal{F}_{\mathcal{A}}$, we can have a weaker version as follows.

1.3. Suppose, X be a Banach Space, and, $I \in C^{\prime}$ $\frac{13}{13}$ Suppose, $\frac{X}{2}$ be a Ranach Space, and $I \in C^1$ **Corollary 4.1.3.** Suppose, X be a Banach Space, and, $I \in C^1(X, \mathbb{R})$ be bounded below. Then, \exists a sequence, $\{x_n\}_{n\geq 0}$ in X satisfying,

$$
I(x_n) \to \inf_{x \in X} I(x)
$$
 and, $I'(x_n) \to 0$ in X^* .

Corollary 4.1.3. Suppose, ^X be a Banach Space, and, ^I [∈] ^C1(X, ^R) be bounded below. Then,

ng the weak form of *Ekeland Variational Principle* (Theorem (4.1.2)), a priori given any $\epsilon > 0$, $\exists x_{\epsilon} \in X$ satisfyin *Proof.* Applying the weak form of *Ekeland Variational Principle* (Theorem (4.1.2)), a priori given any $\epsilon > 0$, $\exists x_{\epsilon} \in X$ satisfying,

$$
I(x_{\epsilon}) \le \inf_{X} I + \epsilon
$$

$$
I(x_{\epsilon}) < I(x) + \epsilon ||x - x_{\epsilon}||, \quad \forall \ x \in X, \ x \neq x_{\epsilon}.\tag{4.1}
$$

 $\epsilon = ty, t > 0, y \in X, y \neq 0.$ $y = ty$, $t > 0$, $y \in Y$, $y \neq 0$. $\frac{1}{1-\frac{1$ $\epsilon = tv, t > 0, v \in X, v \neq 0.$ $y, t > 0, y \in \Lambda$ Choose, x = x^ϵ = ty, t > 0 , y ∈ X , y ̸= 0. Choose, $x = x_{\epsilon} = ty$, $t > 0$, $y \in X$, $y \neq 0$. (4.1) yields,

$$
I(x_{\epsilon}) - I(x_{\epsilon} + ty) < \epsilon t ||y||.
$$

Thus, Thus,

$$
\lim_{t \to 0} \frac{I(x_{\epsilon}) - I(x_{\epsilon} + ty)}{t} \le \epsilon ||y|| \implies -I'(x_{\epsilon})(y) \le \epsilon ||y||, \quad \forall y \in X
$$

$$
\implies I'(x_{\epsilon})(y) \le \epsilon ||y|| \ , \quad \forall \ y \in X \quad (\text{ changing } y \text{ to } -y)
$$

Hence, Hence,

$$
||I'(x_{\epsilon})|| = \sup_{\substack{y \in X \\ y \neq 0}} \left\{ \frac{|I'(x_{\epsilon})y|}{||y||} \right\} \le \epsilon.
$$

Now, taking $\epsilon = \frac{1}{n}$, $x_{\epsilon} = x_n$, we thus have,

$$
\inf_X I \le I(x_n) \le \inf_X I + \frac{1}{n} \implies ||I'(x_n)|| \le \frac{1}{n}.
$$

include our desired result using definition of strong convergence

 $\frac{X}{X}$ $\frac{n}{n}$ $\frac{n}{n}$

4.2 Palais-Smale Condition of strong condition of strong convergence.

We begin with a proper definition of the above. Ω ale Condition 4.2.1. Assume X to be a Banach Space, and, I ∈ R, we say to be a Banach Space, and, I ∈ R, we say to be a Banach Space, and, I ∈ R, we say to be a Banach Space, and I ∈ R, we say to be a Banach Space, a t_{max} satisfies the Palais-Small Condition at α (β) if, every sequence α

breviation) in, every sequence $\{x_n\}_{n\geq 0} \subseteq A$ with, $I(x_n) \to C$, $I(x_n) \to 0$ in A , has a convergent subsequence. c ((PS)) as abbreviation) if, every sequence $\{x_n\}_{n\geq 0} \subset X$ with, $I(x_n) \to c$, $I'(x_n) \to 0$ in X^* , has a convergent subsequence. **Definition 42.1** Assume *X* to be a Banach Space, and $I \in C^1(X, \mathbb{R})$. For any $c \in \mathbb{R}$, we say that *I* satisfies the Palais-Smale Cor **Definition 4.2.1.** Assume X to be a Banach Space, and, $I \in C^1(X, \mathbb{R})$. For any $c \in \mathbb{R}$, we say that, I satisfies the Palais-Smale Condition at

If in fact, *I* satisfies $(PS)_c$ at every $c \in \mathbb{R}$, then, we conclude that, *I* indeed satisfies the Palais-Smale Condition ((*PS*) as in short).

Theorem 4.2.1. *X* be a Banach Space, and, $I \in C^1(X, \mathbb{R})$ is bounded below. Furthermore, let, I satisfy $(PS)_c$, where, $c = \inf_{M}$. Then, $\exists x0$ ∈ *X satisfying*, X , where, c $\frac{X}{X}$

$$
I(x_0) = \inf_{x \in X} I(x) \text{ and, } I'(x_0) = 0.
$$

ary (4.1.3) implies that, ∃ a sequence ${x_n}_{n \geq 0}$ ⊂ X such that, $I(x_n) \to \inf I = c$ (say), and, $I'(x_n) \to 0$ as $n \to \infty$. Since, i subsequence, $\{x_{n_k}\}_{k\geq 0}$ of $\{x_n\}_{n\geq 0}$ and $x_0 \in X$ satisfying, $x_n \to x_0$ in X. *Proof.* Corollary (4.1.3) implies that, \exists a sequence $\{x_n\}_{n\geq 0} \subset X$ such that, $I(x_n) \to \inf Y = c$ (say), and, $I'(x_n) \to 0$ as $n \to \infty$. Since, I satisfies the $(PS)_{c}$, \exists a subsequence, $\{x_{n_k}\}_{k\geq 0}$ of $\{x_n\}_{n\geq 0}$ and $x_0 \in X$ satisfying, $x_n \to x_0$ in X .

Now, $I \in C^1(X, \mathbb{R})$ and $I' \in C(X, X^*)$ both being continuous, we shall obtain,

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 19 \mathcal{L} = 2024 both being continuous, we shall obtain being continuous, we shall obtain being continuous, we shall obtain, we shall obt

$$
I(x_0) = \lim_{k \to \infty} I(x_{n_k}) = c,
$$

 $\overline{}$ in fact, Isatisfies (P $\overline{}$)c at every conclude that, I indeed satisfies that, I indeed satisfies the s

Now, I ∈ C1(X, R) and I′ ∈ C(X, X∗) both being continuous, we shall obtain, we shall obtain, we shall obtain,

 \mathbb{R} a sequence \mathbb{R} a sequence \mathbb{R} such that, I(xn) → information \mathbb{R} (x, a) as $a \in \mathbb{R}$ and, $rac{1}{2}$ $I_{i=1}$ and,

$$
I'(x_0) = \lim_{k \to \infty} I'(x_{n_k}) = 0.
$$

and the result holds true. \blacksquare 4.2.2.2. \blacksquare Ω and the result bolds true P_{max} and, $\frac{1}{2}$ and $\frac{1}{2}$ such that, $\frac{1}{2}$ and $\frac{1}{2}$ such that, $\frac{1}{2}$

Corollary 4.2. Suppose, $\Omega \subset \mathbb{R}^n$ be bounded, and, $1 \le a \le p \le 1$. For every $\lambda \in \mathbb{R}$, we define, $I : H^1$, $(\Omega) \to \mathbb{R}$ as, **2.2.** Suppose, $\Omega \subset \mathbb{R}^n$ be bounded, and, $1 \le q \le p \le \cdots$. For every $\lambda \in \mathbb{R}$, we define, $I : H^1_{\mathfrak{g}}(\Omega) \to \mathbb{R}$ as, \overline{C} $\begin{bmatrix} 1 & c & I & H & Q \\ I & H & H & Q \end{bmatrix}$ **2.2.** Suppose, $\Omega \subset \mathbb{R}^n$ be bound **2.2.** Suppose, $\Omega \subset \mathbb{R}^n$ be bounded, and, $1 \le q \le p \le \cdots$ For every $\lambda \in \mathbb{R}$, we define, $I : H^1_{\ 0}(\Omega) \to \mathbb{R}$ as, **Corollary 4.2.2.** Suppose, $\Omega \subset \mathbb{R}^n$ be bounded, and, $1 \le q < p < \dots$ For every $\lambda \in \mathbb{R}$, we define, $I : H^1_{\mathbb{Q}}(\Omega) \to \mathbb{R}$ as, $\binom{n}{2}$. For every $\binom{n}{2}$ **Corollary 4.2.2.** Suppose, $\Omega \subset \mathbb{R}^n$ be bounded, and, $1 \le q \le p \le$ For every $\lambda \in \mathbb{R}$, we define, $I : H^1_{\ 0}(\Omega) \to \mathbb{R}$ as, **Corollary 4.2.2.** Suppose, $Q \subseteq \mathbb{R}^n$ be bounded, and, $1 \le a \le p \le q$. For every $\lambda \in \mathbb{R}$, we define, $I : H^1(\Omega)$

 \overline{C} and, I \overline{C} is bounded below. Furthermore, and, I \overline{C} is bounded by \overline{C}

$$
I(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} - \frac{\lambda}{2} \int_{\Omega} |u|^{q+1}
$$
(4.2)
Then, I satisfies the (PS) condition.

(xn) → 0 in X×, has a convergent subsequence. A convergent subsequence.

Then, I satisfies the (PS) condition.

 ${x_0}$ and ${x_0}$ and ${x_0}$ in ${$

{xn}n≥⁰ ⊂ X with, I(xn) → c , I′

Corollary 4.3. Let, $p = \frac{(n+2)}{(n-2)}$, and, I be defined as in (4.2). Then, I satisfies (PS)_c for every $c \in (-\infty, \frac{1}{n}S^{n/2})$, where, $\begin{pmatrix} n+2 \\ n \end{pmatrix}$ conditions the conditions the conditions of $\begin{pmatrix} n+2 \\ n \end{pmatrix}$ $(n-2)$ \mathcal{L} **C.3.** Let, $p = \frac{(n+2)}{(n-2)}$, and, I be defined as in (4.2). Then, I satisfies (PS) for every $c \in (-\infty, \frac{1}{n}S^{n/2})$, where, −∞, ¹ **C.3.** Let, $p = \frac{(n+2)}{2}$, and, I be defined as in (4.2). Then, I satisfies (PS) for every $c \in (-\infty, \frac{1}{2}S^{n/2})$, where, $\boldsymbol{\mu}$. Let, $p - \frac{n-2}{n-2}$, and, the defined for given $c \in (-\infty, 1 \text{ sn}/2)$ **Corollary 4.2.3.** Let, $p = \frac{(n+2)}{(n-2)}$, and, I be defined as in (4.2). Then, I satisfies (PS)_c for every $c \in (-\infty, \frac{1}{n}S^{n/2})$, where, $\binom{n-1}{k}$ **Corollary 4.2.3.** Let, $p = \frac{(n+2)}{(n-2)}$ Coronary 4.2.3. Let, $p -$ **Corollary 4.2.3.** Let, $p = \frac{(n+2)}{(n-2)}$, and, I be defined as in (4.2). Then, I satisfies (PS) **Coronary 4.2.3.** Let, $p - \frac{n-2}{n-2}$, and, I be defined as in (4.2). Then, I satisfies (1.3)_c for **Corollary 4.2.3.** Let, $p = \frac{(n+2)}{(n-2)}$, and, I be defined as in (4.2). Then, I so **1.3.** Let, $p = \frac{p-2}{(n-2)}$, and, I be defined as in (4.2). Then, I satisfies $(PS)_{c}$ for every $c \in (-\infty, \frac{1}{n}S^{n/2})$, where,

$$
S = \inf \left\{ \int_{\Omega} |\nabla u|^2 \ : \ u \in H_0^1(\Omega) \ and, \int_{\Omega} |u|^{2n/(n-2)} = 1 \right\}.
$$

See defined above is termed as the Best Sobolev Constant Moreover, I does not satisfy (PS), where $c = \frac{1}{n} S^{n/2}$.

_{olev} Ω
ev Constant. Moreover, I does not satisfy (PS)_c, where, $c=$ bove is termed as the Best Sobolev Constant. Moreover, I does not satisfy (PS)_c, where, $c=\frac{1}{n}S^{n/2}$. S as defined above is termed as the Best Sobolev Constant. Moreover, I does not satisfy $(PS)_{c}$, where, $c=\frac{1}{n}S^{n/2}$ $\sum_{i=1}^N$ as the Best Sobolev Constant. Moreover, I week not satisfy $\sum_{i=1}^N$ metre, $c = \frac{1}{n}$ d above is ter ed as the Best Sobolev Constant. Moreover, I does not s rmed as the 11
st Sobolev Constant. Moreover, 1 $\frac{N}{2}$ ed above is termed as the Best Sobolev Constant. Moreover, I does not satisfy (PS)_c, where, $c = \frac{1}{n} S^{n/2}$. define, I : H¹ ⁰ (Ω) → R as, I′ (x0) = limk→∞ ^I′ (xn^k)=0. $\frac{S_4}{S_4}$
See defined shows is towned so the Best Scholar Constant, Menecusy, I does not estimbly (BS), where $s = \frac{1}{2}$ CM/ x_n as a constant $x_n > 0$ is termed as the Dest Socotor Constant We have the second to show the sequence $\sum_{i=1}^n$ and $\sum_{i=1}^n$ and $\sum_{i=1}^n$

S as defined above is termed as the Best Sobolev Constant. Moreover, I does not satisfy hove that I satisfies $(PS)_c$ for $c = \frac{1}{n}S^{n/2}$, where S is the Best Sobolev Constant, we proceed as follows: *Proof.* • To prove that *I* satisfies (PS) _c for $c = \frac{1}{n}S^{n/2}$, where *S* is the Best Sobolev Constant, we proceed as follows: $\frac{1}{2}$ and $\frac{1}{2}$ S as defined above in the Best Sobolev Constant \sum_{c} is the Best Sobolev Constant, we give Ω Ω Ω *Proof.* • To prove that *I* satisfies (PS) for $c = \frac{1}{n}S^{n/2}$, where *S* is the Best Sobolev Constant, we proceed as follows: *Proof.* • To prove that I satisfies $(PS)_c$ for $c = \frac{1}{n}S^{n/2}$, where S is the Best Sobolev Constant, we proceed as follows:

 $\mathcal{S}^{n/2}$ let $\mathcal{S}_u \subset H^1$ (O) be Given $c = \frac{1}{n}S^{n/2}$, let $\{u_n\} \subset H^1(\Omega)$ be a Palais-Smale sequence at level c. By definition, this means that $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. ri I as $n \rightarrow \infty$. $\overline{2}$ ¹ Ω $\lceil \text{e} \rceil \rceil$ \overline{a} be a Palais-S ale sequence at level c . By definition Given $c = \frac{1}{n} S^{n/2}$, let $\{u_n\} \subset H^1(\Omega)$ be a Palais-Smale sequence at level c. By definition, this means that $I(u_n) \to c$ and I' as $n \to \infty$.

We aim to show that every Palais-Smale sequence at level c has a convergent subsequence. To do so, we will use the Mountain Pass Subham De 27 III de
De 27 III de 27 III \mathbf{u} Theorem. Subham De 27 Indian De 27 I o show that every Palais-Smale sequence at level c has a convergent subsequence. To do so, we will use the M
M ^q+1 (4.2)

The *Mountain Pass Theorem* states that if *I* satisfies certain conditions, including the Palais-Smale condition and coercivity, then it possesses a critical point at every level below the Mountain Pass value. н.
... Fire *Mountain* T ass Theorem states that HTT satisfies certain conditions, including sesses a critical point at every level below the Mountain Pass value. $\ddot{}$ The Mountain Pass Theorem states that if I satisfies certain conditions, including the Palais-Smale condition and coercivity, the

Now, since $c = \frac{1}{n} S^{n/2}$, it implies that c is below the Mountain Pass value *γ*. Therefore, by the Mountain Pass Theorem, every Palais-Smale sequence at level c has a convergent subsequence converging to a minimum of *I*. Ω is below the Mountain Pass value γ . Therefore, by the M $\frac{1}{2}$ converged is below the Mountain Pass value γ . Therefore, by the Ment subsequence converging to a minimum of I . Now, since $\epsilon = \frac{1}{n}$, ϵ , it implies that c is below the Mountain Pass value γ . There, also Smale sequence at level c has a convergent subsequence converging to a minimum .
oun Now, since $c = \frac{1}{n} S^{n/2}$, it implies that c is below the Mountain Pass value γ . Therefore, by the Mou Now, since $c = \frac{1}{n} S^{n/2}$, it implies that c is below the Mountain Pass value γ . Therefore, by the Mountain Pas

Subham De 27 IIT Delhi, India Hence, *I* satisfies (PS) _c for $c = \frac{1}{n}S^{n/2}$. \mathcal{L} S _c for $c = \frac{1}{n}$ $c = \frac{1}{n} S^{n/2}$ T^2 $\frac{1}{2}$ $\frac{m}{2}$ \sum_{c} for $c = \frac{1}{n}$

• To prove that (PS) _c fails for $C = \frac{1}{n} S^{n/2}$, where *S* is the Best Sobolev Constant, we construct a Palais-Smale sequence $\{u_n\} \subset H^1$ ₀ The prove that $(P, S)_c$ can be a convergent subsequence (Ω) at level *c* that does not have a convergent subsequence • To prove that (PS) _c fails for $c = \frac{1}{n}S^{n/2}$, where S is the Best Sobolev Constant, we construct a Palais-S fail $c = \frac{1}{n} S^{n/2}.$ \mathbf{u} \int_0^{∞} , where *S* is the Best Sobolev Consta \mathcal{S} as defined as the Best Sobolev Constant. Moreover, I does not satisfy \mathcal{S} fails for $c =$ \ddotsc for fixed \ddotsc $\frac{1}{2}$ and $\frac{1}{2}$ are constructed $\frac{1}{2}$ as follows:

Given $c = \frac{1}{n}S^{n/2}$, we construct a Palais-Smale sequence $\{u_n\}$ as follows: Define, Given $c = \frac{1}{2} S^{n/2}$, we construct a Palais-Smale sequence $\{u_n\}$ as follows: Define, Given $c = \frac{1}{n} S^{n/2}$, we construct a

$$
u_n(x) = \sqrt{\frac{2}{\lambda_1(\Omega)}} \sin(\lambda_n x_1) \sin(\lambda_n x_2) \cdots \sin(\lambda_n x_n),
$$

where λ_n is the *n*-th eigenvalue of $-\Delta$ with Dirichlet boundary conditions. \overline{a} where $λ_n$ is the *n*-th eigenvalue of $-Δ$ with Dirichlet boundary conditions.

Each function u_n is an eigenfunction of $-\Delta$ with Dirichlet boundary conditions, normalized such that $||u_n|| = 1$. Therefore, $I(u_n)$ achieves the desired level $a = \frac{1}{2} \sin/2$ the desired level $c = \frac{1}{n} S^{n/2}$. $\frac{1}{n}$

I_{(xn}k) → limk→∞ I(xnk) → c,

I(xo) → limk→∞
I(xnk) = c,

By the properties of eigenfunctions, $I'(u_n) = u_n - \lambda_1(\Omega)u_n = (1 - \lambda_1(\Omega))u_n$. Since $\lambda_1(\Omega) > 0$, $I'(u_n) \to 0$ as $n \to \infty$. \mathbb{R} s \mathbb{Z} $\overline{}$ $\overline{}$. Sin $e \lambda_1(\Omega) > 0, I'(u_n) \to 0 \text{ as } n \to \infty$ every ^c [∈] � −∞, ¹ nSn/² , where, By the properties of eigenfunctions, $I'(u_n) = u_n - \lambda_1(\Omega)u_n = (1 - \lambda_1(\Omega))u_n$. Since $\lambda_1(\Omega) > 0$, $I'(u_n) \to 0$ as $n \to \infty$. $D > 0$ $I'(u) > 0$ os $n \to \infty$ $a_n = u_n - \lambda_1(\Omega) u_n = (1 - \lambda_1(\Omega)) u_n$. Since $\lambda_1(\Omega)$

However, the sequence $||u_n||$ does not converge, as it remains constant for all n. This lack of convergence implies that there is no convergent subsequence, violating the Palais-Smale condition. S as defined as the Best Sobolev Constant. More over, I does not satisfy \mathcal{L} s it remains constant for all n. This lack of convergence is dition. However, the sequence $||u||$ do ale condition. \mathcal{L}^{max} $\frac{1}{\sqrt{2}}$

Hence, we've found a Palais-Smale sequence $\{u_n\}$ at level $c = \frac{1}{n} S^{n/2}$ that does not have a convergent subsequence. Therefore, I fails the Palais-Smale condition at $c = \frac{1}{n}S^{n/2}$. quence $\{u\}$ at level c Ω $\mathcal{A} = \mathcal{A} \cup \mathcal{A}$. The contraction of the contraction $\mathcal{A} = \mathcal{A} \cup \mathcal{A}$ ance $\{u\}$ at level $c = \frac{1}{n} S^{n/2}$ that does not Corollary 4.2.2. Suppose, 2.2.2. Suppose, 2.2.2. Suppose, and, 1 ≤ qualitative bounded, and 1 ≤ qualitative bounded, and 1 ≤ quali

• Given the functional $I(u)$ defined as: α as:

$$
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\lambda_1(\Omega)}{2} \int_{\Omega} u^2
$$

where λ_n is the *n*-th eigenvalue of $-\Delta$ with Dirichlet boundary conditions |∇u| : ^u [∈] ^H¹ he *n*-th eigenvalue of $-\Delta$ with Dirichlet boundary conditions

Each function u is an eigenfunction of $-\Delta$ with Dirichlet boundary conditions, normalized such that $||u|| = 1$. Th the desired level $c = \frac{1}{n}S^{n/2}$. Each function u_n is an eigenfunction of $-\Delta$ with Dirichlet boundary conditions, normalized such that $||u_n|| = 1$. Therefore, $I(u_n)$ achieves nSn/² μ_n is an eigenfunction of $-\Delta$ with Dirichlet boundary conditions, normalized such that $\|\mu_n\| = 1$. Therefore, $I(\mu_n)$
vel $c = \frac{1}{n} S^{n/2}$. n^{\sim}

 \overline{S} $U_n = (1 - \lambda_1(\Omega))u_n$. Since $\lambda_1(\Omega) > 0$, $I'(u_n) \to 0$ as By the properties of eigenfunctions, $I'(u_n) = u_n - \lambda_1(\Omega)u_n = (1 - \lambda_1(\Omega))u_n$. Since $\lambda_1(\Omega) > 0$, $I'(u_n) \to 0$ as $n \to \infty$. \mathbb{R} s \mathbb{R} \overline{c} , Sii $\lambda_1(\Omega) > 0, I'(u_n) \to 0 \text{ as } n \to \infty$ $n[−]$ u_n λ_1 (22) u_n (1 λ_1 (22)) u_n , since λ_1 (22) λ 0, 1 (u_n) λ 0 de

 $\sum_{n=1}^{\infty}$ in sequence $\frac{a_n}{\|a_n\|}$ abostor converge, as it remains constant for an n. This fact of convergence implies to However, the sequence $||u_n||$ does not converge, as it remains constant for all *n*. This lack of convergence implies that there is no convergent subsequence, violating the Palais-Smale condition. gent subsequence, violating the Palais-Smale condition. S as defined as the Best Sobolev Constant. More over, I does not satisfy \mathcal{L} uence $||u_n||$ does not converge, as it remains con
violating the Palais-Smale condition. \mathbf{r} t for all *n*. This la k of c

Hence, we've found a Palais-Smale sequence $\{u_n\}$ at level $c = \frac{1}{n}S^{n/2}$ that does not have a convergent subsequence. Therefore, I fails the Palais-Smale condition at $c = \frac{1}{n}S^{n/2}$. a Palais-Smale sequence $\{u_n\}$ at level $c = \frac{1}{n} S^{n/2}$ that does not have a convergent subsequence. Therefore dition at $c = \frac{1}{n} S^{n/2}$

Given the functional $I(u)$ defined as:

$$
I(u)
$$
 defined as:

$$
I(u)=\frac{1}{2}\int_{\Omega}|\nabla u|^2-\frac{\lambda_1(\Omega)}{2}\int_{\Omega}u^2
$$

where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $-\Delta$ on Ω , we aim to derive the Best Sobolev Constant S, defined as: $\displaystyle{\operatorname{rst}$ Dirichlet eigenvalue of $-\Delta$ on Ω , we aim to derive the Best Sobolev Constant S, def $2n/2$ \mathbf{E} the Euler-Lagrange equation, the critical points of \mathbf{E}

$$
S = \inf \left\{ \int_{\Omega} |\nabla u|^2 : u \in H_0^1(\Omega), \int_{\Omega} |u|^{2n/(n-2)} = 1 \right\}
$$

 $J(u)$ subject to the constraint satisfy: $Ω$ $JΩ$ $JΩ$ J Consider the functional J(u) = Consider the functional $J(u) = \int_{\Omega} |\nabla u|^2$ subject to the constraint $\int_{\Omega} |u|^{2n/(n-2)} = 1$. By the Euler-Lagrange equation, the critical points of \mathcal{L}_{eff}

$$
-\Delta u + \lambda |u|^{2n/(n-2)-2}u = 0
$$

ge multiplier. Let *u* be a nontrivial solution of the above equation. By scate Then, by the variational characterization of $\lambda_1(\Omega)$, we have: where λ is a Lagrange multiplier. Let *u* be a nontrivial solution of the above equation. By scaling, we may assume that $||u||_{L^{2n/(n-2)}} = 1$.
Then, by the variational characterization of $\lambda_1(\Omega)$, we have:

$$
\lambda_1(\Omega) = \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^2}
$$

Since $||u||_{L^{2n/(n-2)}} = 1$, we have $\int_{\Omega} |u|^{2n/(n-2)} = 1$, implying that $\int_{\Omega} |u|^2$ achieves its maximum. Thus, $\lambda_1(\Omega)$ is the smallest eigenvalue. There- \sim 1. \sim 2 − λ1(Ω)
2 Ω)
2 Ω) where $\mathcal{L}_{\mathcal{A}}$ is a Lagrange multiplier. Let us a non-trivial solution of the above equation. By Since $\|\mathbf{u}\|_{L^{2m(n-2)}}$ 1, we have \mathbf{v}_{Ω} [*u*| 1, implying that \mathbf{v}_{Ω} [*u*| actively is the maximum. Thus, $\mathbf{v}_{\mathcal{I}}(\mathbf{x}_2)$ is the smallest eigenvalue fore, the Best Sobolev Constant *S* is equal to the s assume that $\frac{3\sqrt{2}}{2}$ = 1. Then, by the variational characterization by the variational characterization of $\frac{3\sqrt{2}}{2}$ = 1.

<u></u>

Hence, $S = \lambda_1(\Omega)$. This completes the proof.

4.2.4. Suppose, $\lambda_1(\Omega)$ be the first Dirichlet Eigenvalue of $-\Delta$. Define, $I: H^i_{0}$ $Define, I$ **Proposition 4.2.4.** Suppose, $\lambda_1(\Omega)$ be the first Dirichlet Eigenvalue of $-\Delta$. Define, $I: H^1_{0}(\Omega) \longrightarrow \mathbb{R}$ as,

S is equal to the first Dirichlet eigenvalue $\mathcal{L}(\mathcal{L})$ of $\mathcal{L}(\mathcal{L})$ of $\mathcal{L}(\mathcal{L})$ of $\mathcal{L}(\mathcal{L})$

Since [∥]u∥L2n/(n−2) = 1, we have

Curr Res Stat Math, 2024 2024 Hence, S = λ1(2). This complete the proof. This complete the proof. <u></u>

Ξ

$$
I(u):=\frac{1}{2}\int\limits_{\Omega}|\nabla u|^{2}-\frac{\lambda_{1}(\Omega)}{2}\int\limits_{\Omega}u^{2}.
$$

Then, I does not satisfy Palais Smale Condition at the point 0.

Proof. Given the functional $I(u)$ defined as: Proof. Given the functional I(u) defined as:

0 (Q) −
| 2 (Q) −→ R as, R a

 \mathbf{H}

$$
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\lambda_1(\Omega)}{2} \int_{\Omega} u^2
$$

 $\frac{2 J_{\Omega}}{J_{\Omega}}$ is first Dirichlet eigenvalue of $-\Delta$ on Ω , we aim to demonstrate that *I* does not satisfy the Palais-Smale condition not satisfy the Palais-Smale condition at the point 0. where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $-\Delta$ on Ω , we aim to demonstrate that I does not satisfy the Palais-Smale condition at the point 0 To do so, we construct a sequence {un} ⊂ ^H¹ ⁰ (Ω) such that I(un) → 0, ∥un∥→∞, and point 0.

struct a sequence $\{u_n\} \subset H^1_{\ 0}(\Omega)$ such that $I(u_n) \to 0$, $||u_n|| \to \infty$, and $I'(u_n) \to 0$ as $n \to \infty$. $\sum_{n=1}^{\infty} a_n$ construct a sequence $\{u_n\}$ To do so, we construct a sequence $\{u_n\} \subset H^1_{\mathfrak{g}}(\Omega)$ such that $I(u_n) \to 0$, $||u_n|| \to \infty$, and $I'(u_n) \to 0$ as $n \to \infty$.

function, $\sum_{i=1}^{n}$ constant the function, Consider the function, Consider the function, Consider the function, Consider the function,

$$
u_n(x) = \left(\sqrt{\frac{2}{\lambda_1(\Omega)}}\right)^{\frac{1}{n}} \sin(\lambda_n x_1) \sin(\lambda_n x_2) \cdots \sin(\lambda_n x_n),
$$

 (x_1, x_2, \ldots, x_n) represents the spatial coordinates in Ω , and λ_n is the *n*-th eigenvalue of $-\Delta$ with Dirichlet boundary co Now, let's justify the properties: Δ and μ and Δ $\alpha_2, \ldots, \alpha_n$, Dirichlet boundary conditions. x_2, \ldots, x_n) represents the spatial coordinates in Ω , and λ_n is the *n*-th eigenvalue of $-\Delta$ with Dirichlet boundary conditions. where $x = (x_1, x_2, \dots, x_n)$ represents the spatial coordinates in Ω , and λ_n is the *n*-th eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

 \mathcal{L} is properties: $\frac{1}{\sqrt{2}}$ • Convergence of $I(u_n) \to 0$: Substituting un into $I(u)$, we have:

$$
I(u_n) = \frac{1}{2}\lambda_n - \frac{\lambda_1(\Omega)}{2} \to 0
$$

 $I(u_n) = \frac{1}{2}\lambda_n - \frac{\lambda_1(s)}{2} \to 0$
as $n \to \infty$, since λ_n is the *n*-th eigenvalue and $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ on Ω . as n is the n-th eigenvalue and $\frac{1}{2}$ is the first eigenvalue of $\frac{1}{2}$

as n is the n-th eigenvalue and α is the first eigenvalue of α on α . $\sum_{n=1}^{\infty}$ ung norm of u_n is given by: • Divergence of $||u_n|| \to \infty$: The norm of u_n is given by:

$$
||u_n|| = \left(\sqrt{\frac{2}{\lambda_1(\Omega)}}\right)^{\frac{1}{n}} \to \infty
$$

as $n \rightarrow \infty$.

 $\lambda \sim 0.71$ Convergence of $I'(u_n) \to 0$: The derivative of $I(u)$ at un is given by:

$$
I'(u_n) = (1 - \lambda_1(\Omega))u_n \to 0
$$

 α , since α , since α $a_2 > 0.$ $a_2 > 0.$ as $n \to \infty$, since $\lambda_1(\Omega) > 0$.

as the conditions frace $\{u_n\}$ satisfies the conditions frace Therefore, the sequence $\{u_n\}$ satisfies the conditions for the Palais-Smale condition to fail at the point 0. Thus, the functional I does not satisfy $(PS)_0$. satisfy $(PS)_{0}$.

i. Consider a Banach Space X and, I \in C¹(X, R). Assume, $\alpha = \lim_{||u|| \to \infty} I(u)$ 6 $\rightarrow \infty$, $I(u_n) \rightarrow \alpha$ and, $I'(u_n) \rightarrow 0$.
 Proof. A priori it is given to us that, $I \in C^1(X, \mathbb{R})$ on the Banach space X, with $\alpha = \lim_{\|u\| \to \infty} I(u) \in \mathbb{R}$. α and, $\Gamma(\mathbf{u}_n) \to 0$. α and, $I'(\mathbf{u}_n) \to 0$. $\mathbb{I}^{|\mathbf{a}_n| \to \infty}$ **Lemma 4.2.5.** Consider a Banach Space X and, $I \in C^1(X, \mathbb{R})$. Assume, $\alpha = \lim_{\|u\| \to \infty} I(u) \in \mathbb{R}$. Then, \exists a sequence, $\{u_n\}_{n \geq 0} \subset X$ satisfying, $\|u_n\|$ $\rightarrow \infty$, $I(u_n) \rightarrow \alpha$ and, $I'(u_n) \rightarrow 0$.

 $||u||_{\infty}$

i it is given to us that, $I \in C^1(X, \mathbb{R})$ on the Banach space X, with $\alpha = \lim I(u) \in \mathbb{R}$. \mathcal{L} Let $\epsilon > 0$ and $u_0 \in X$. Then, Ekeland's Variational Principle says that ∃ $u \in X$ such that: I(u) ∈ R. Let ϵ > 0 and $u_0 \in \Lambda$. Then, Ekeland's Variational Principle says that ∃ $u \in \Lambda$ such that:

1. $I(u) \leq I(u_0)$,

lim

- 2. $||u u_0|| < \epsilon$,
- 3. $I(u) < I(v) + \frac{1}{2\epsilon} ||u v||^2$ for all $v \neq u$.

Now, we try to construct a sequence $\{u_n\}$ which'll satisfy the desired conditions. Choose $u_0 \in X$ arbitrarily and $\epsilon = 1$. By Ekeland's Variational Principle, there exists $u_1 \in X$ satisfying conditions (1), (2), and (3) with $\epsilon = 1$.

conditions (1), α and α and α and (2) with α and (3) with α Now, recursively construct the sequence $\{u_n\}$ as follows: For $n \ge 2$, apply Ekeland's Principle with $\epsilon = 1/n$ and $u_0 = u_{n-1}$. This gives us un satisfying conditions (1), (2), and (3) with $\epsilon = 1/n$. $n = 1$. This gives us under the unit satisfying conditions (1), and (3) with α = 1, and (3) with α = un satisfying conditions (1), (2), and (3) with $\epsilon = 1/n$.

By our construction, the sequence $\{u_n\}$ satisfies the following properties:

- $I(u_n) \leq I(u_{n-1})$ for all *n*, implying that $\{I(u_n)\}\$ is a decreasing sequence.
- $||u_n u_{n-1}|| < 1/n$ for all *n*, so $\{u_n\}$ is a Cauchy sequence.
- By the completeness of *X*, $\{u_n\}$ converges to some $u \in X$.

Furthermore, since $\{I(u_n)\}\$ is decreasing and bounded below by α , it converges to α . Also, since I is continuously differentiable, $I'(u_n)$ converges to *I'* (*u*) = 0 as $n \rightarrow \infty$.

Assume if possible that, $||u_n|| \to \infty$. Then, there exists $M > 0$ such that $||u_n|| \leq M$ for all *n*. But then $\lim_{n \to \infty} I(u_n) = -\infty$, contradicting the assumption that α is finite. Thus, $||u_n||$ must tend to infinity.

Therefore, we have constructed a sequence $\{u_n\}$ satisfying all the desired properties using Ekeland's Variational Principle. Thus the \mathbf{F} and \mathbf{F} is complete. Thus the principle is complete. Thus the principle. Thus the proof is complete. proof is complete.

As a consequence of Lemma $(4.2.5)$, we can conclude the following. $\mathcal{A}^{\text{max}}_{\text{max}}$ \mathcal{C} in fact observe an application of Ekeland Variation of Ekeland Variation \mathcal{C}

Proposition 4.2.6. If $I \in C^1(X, \mathbb{R})$ is bounded below, and satisfies the (PS) condition, then, I is coercive. We can in fact observe an application of Ekeland Variation of Ekeland Variation of \mathbb{R}^n

 \mathbb{R}^2

Proposition 4.3 Applications \overline{m}

We can in fact observe an application of *Ekeland Variational Principle* to derive critical point(s) at the minimax level. ons
t observe an application of *Ekeland Variational Principle* to derive critical point(s) at the minimax level. $\frac{1}{1}$

Theorem 4.3.1. (Brezis Theorem) Given a Banach Space X and, $I \in C^1(X, \mathbb{R})$, assume K to be a compact metric space. Moreover, let, K_0 $\subset K$ *be a closed set, and,* $p_{0}: K_{0} \rightarrow X$ *be a continuous function.* to be a compact metric space. The angle S_{C} and $I \in C((V, \mathbb{R})$ is a compact with an and, M_{C} $\frac{c_1}{c_2}$ continuous function. $\frac{c_1}{c_2}$ or $\frac{c_2}{c_1}$ **.1.** (Brezis Theorem) Giv μ , and, μ

Define,

$$
\Gamma := \{ p \in C(K, X) \ni p |_{K_0} = p_0 \}
$$

 $\mathcal{L}(\mathcal{L})$

 $and,$ *and,*

$$
c := \inf_{p \in \Gamma} \max_{\xi \in K} I(p(\xi)) = \inf_{p \in \Gamma} \max_{x \in p(K)} I(x). \tag{4.3}
$$

 \mathcal{A} ALso, suppose that, *ALso, suppose that,* \overline{A}

$$
\max_{\xi \in K} I(p(\xi)) > \max_{\xi \in K_0} I(p_0(\xi)) \qquad \forall \ p \in \Gamma.
$$
\n(4.4)

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 23 $\frac{1}{2}$ Math, $\frac{2024}{2}$

 $\begin{array}{ccccc} F^1 & \rightarrow & \begin{array}{ccccc} \begin{array}{ccccc} \end{array} & \$ *with*, $i(x_0) = c$ and, $I'(x_0) = 0$. Then, \exists a sequence, $\{x_n\}_{n\geq 0} \subset X$ satisfying, $I(x_n) \to c$ and, $I'(x_n) \to 0$ in X^* as $n \to \infty$. Furthermore, if I satisfies $(PS)_{\infty}$, then, $\exists x_0 \in X$ $\mathcal{L}(\mathcal{C})$

Important to mention that, the proof mainly hinges upon the following result.

Lemma 4.3.2. (Pseudo-Gradient Lemma) Assume any metric space Y, and X to be a Banach Space with, $F \in C(Y, X^*)$. Given any $\sigma > 0$ \exists a function $h: Y \rightarrow X$ locally Lipschitz such that $\forall y \in Y$ 0, ∃ *a function,* $h: Y \rightarrow X$ *, locally Lipschitz such that,* $\forall y \in Y$,

- $||h(y)||_X \leq 1$.
- $\langle F(y), h(y) \rangle \ge ||F(y)||_{X^*} \sigma.$

Brezis Theorem can also be considered as a generalization of the Mountain-Pass Theorem introduced by Ambrosetti and Rabinowitz. A as a generanzation of the mountain I ass Theorem intodaced by *innorosem* and *habh*

introduced by Ambrosetti and Rabinowitz. Theorem 4.3.3. (Mountain Pass Theorem) A priori under the assumptions that, X be a $\frac{1}{2}$ Moreover, we consider the following condition, **Theorem 4.3.3.** (Mountain Pass Theorem) A priori under the assumptions that, X be a Banach Space and, $I \in C^1(X, \mathbb{R})$ satisfies (PS).
Moreover we consider the following condition

$$
\exists R > 0 \text{ and, } e \in X \text{ such that, } ||e|| > R \text{ and, } b = \inf_{x \in \partial B_R(0)} I(x) > \max\{I(0), I(e)\}. \tag{4.5}
$$

Then, $\exists x \in X$ satisfying, $I'(x \infty) = 0$ and, $I(x \infty) = c \ge b$, where, we can in fact characterize c as, *Then*, $\exists x_0 \in X$ *satisfying*, $I'(x_0) = 0$ *and*, $I(x_0) = c \ge b$, *where, we can in fact characterize c as*, X satisfying, $I'(x_0) = 0$ and, $I(x_0) = c \ge b$, where, we can in fact chard

$$
c = \inf_{p \in \Gamma} \max_{t \in [0,1]} I(p(t)).
$$

 \mathcal{L} is a subhaming \mathcal{L} in \mathcal{L} is a subhaming \mathcal{L} is a subhaming \mathcal{L} is a subhaming $\mathcal{$ *such that,*

$$
\Gamma = \{ p \in C((0,1], X) : p(0) = 0 , p(1) = e \}.
$$

ri from the definition, we have, $c < \infty$. Thus, for every $p \in \Gamma$, $p([0, 1]) \cap \partial B_R(0) \neq \phi$. As a result, e, c < ∞. Thus, for every $p \in \Gamma$, $p([0, 1]) \cap \partial B_{\kappa}(0) \neq \phi$. As a result, *Proof.* A priori from the definition, we have, *c* < ∞. Thus, for every $p \in \Gamma$, $p([0, 1]) \cap \partial B_R(0) \neq \emptyset$. As a result,

$$
\max_{t \in [0,1]} I(p(t)) = \max_{x \in p[0,1]} I(x) \ge \inf_{x \in \partial B_R(0)} I(x) = b \tag{4.6}
$$

 δ Therefore, $c \ge b$. To establish the existence of critical point, we need to utilize the hypothesis of Brezis Theorem. Let, $K = [0, 1]$, $K_0 = (0, 1)$ T_{max} the existence of critical point, we need to utilize the hypothesis of utilization of utilization of \mathbb{R} θ and, $P_0(1)$, c. $-\sigma$ and, $p_0(1) - e$. $\{0, 1\}$, $p_0(0) = 0$ and, $p_0(1) = e$.

 T_{eff} b. To establish the existence of contract point, we need to utilize the hypothesis of \mathcal{L} emains for us to verify the condition (4.4). Although, from (5.1), we obtain, Now then it remains for us to verify the condition (4.4). Although, from (5.1), we obtain,

$$
\max_{t \in [0,1]} I(p(t)) \ge \inf_{x \in \partial B_R(0)} I(x) = b > \max\{I(0), I(e)\}.
$$

that that, $\exists x_0 \in X$ satisfying, $I(x_0) = c$ and, $I'(x_0) = 0$. This compl that, $\exists x_0 \in X$ satisfying, $I(x_0) = c$ and, $I'(x_0) = 0$. This compl Applying *Brezis Theorem*, we conclude that, $\exists x_0 \in X$ satisfying, $I(x_0) = c$ and, $I'(x_0) = 0$. This completes the proof.

I. The statement of the *Mountain Pass Theorem* need not be true in case if *I* does not satisfies (*PS*). **Remark 4.3.4.** The statement of the *Mountain Pass Theorem* need not be true in case if *I* does not satisfies (*PS*).

For example, we can look into the function, $I \in C^1(\mathbb{R}^2, \mathbb{R})$ defined as,

$$
I(x, y) := x^2 - (x - 1)^3 y^2.
$$

SInce, $I(0) = 0$ and, 0 is in fact a *local minima*, we can choose $R > 0$ small enough such that, $b = \inf_{x \in \partial BR(0)} I(x) > 0$. Furthermore, given $e \in \mathbb{R}^2$ $\sum_{i=1}^{n}$ and $\sum_{i=1}^{n}$ $\sum_{i=1}^{n}$ $\sum_{k=1}^{\infty}$ If the conditions of the *Mountain* \mathbb{R}^2 with, $|e| > \mathbb{R}$ and, $I(e) < 0$. It can be observed that, all the conditions of the *Mountain Pass Theorem* are satisfied, except that, I satisfies (*PS*). (*PS*). *x*∈∂*BR*(0)

 $\mathcal{S}^{\text{max}}_{\text{max}}$ and, $\mathcal{S}^{\text{max}}_{\text{max}}$ and, we can choose $\mathcal{S}^{\text{max}}_{\text{max}}$ such that, we can choose $\mathcal{S}^{\text{max}}_{\text{max}}$

 $\overline{}$, $\overline{}$, $\overline{}$ with, $\overline{}$ and, $\overline{\$

 $\mathcal{L}(\mathcal{L}) = \mathcal{L}(\mathcal{L})$ and $\mathcal{L}(\mathcal{L}) = \mathcal{L}(\mathcal{L})$ with, $\mathcal{L}(\mathcal{L}) = \mathcal{L}(\mathcal{L})$ and, if $\mathcal{L}(\mathcal{L}) = \mathcal{L}(\mathcal{L})$

satisfies (P S).

Also, since $I(0) = 0$ and, 0 is indeed the only critical point of *I*, there \sharp any $x_0 \in \mathbb{R}^2$ satifying, $I(x_0) \ge b > 0$ and, $I'(x_0) = 0$. (0) = 0 and, 0 is indeed the only critical point of *I*, there \sharp any $x_0 \in \mathbb{R}^2$ satifying, $I(x_0) \ge b > 0$ and, $I'(x_0) = 0$. Also, since $I(0) = 0$ and, 0 is indeed the only critical point of I, there \sharp any $x_0 \in \mathbb{R}^2$ satifying, $I(x_0) \ge b > 0$ and, $I'(x_0) = 0$.

Remark 4.3.5. Geometrically speaking, *Mountain Pass Theorem* implies that, if a pair of points in the graph of a function *I* are indeed **Remark 4.3.5.** Geometrically speaking, *Mountain Pass Theorem* implies that, if a pair of points in the graph of a function *I* at separated by a mountain range, then \exists a *mountain pass* containing a critical point of $T_{\rm eff}$, and $T_{\rm eff}$ is $T_{\rm eff}$ indeed a Banach Space X, and I α is $T_{\rm eff}$ indeed a constructional which indeed a $T_{\rm eff}$

Another version of the famous Mountain Pass Theorem can be found in [15].

Theorem 4.3.6. Given a Banach Space X, and $I: X \to \mathbb{R}$ be a C¹ functional which indeed satisfies the (PS) condition. Suppose, S be a closed subset of X which disconnects X. Furthermore, given $x_0, x_1 \in X$ which belong to distinct connected components of $X \setminus S$, if I is bounded below in S, and in fact, the following condition is verified:

$$
\inf_{S} I \ge b \text{ and } \max\left\{I(x_0), I(x_1)\right\} < b \tag{4.7}
$$

 $\mathbf{S} = \mathbf{S} \mathbf{S} \mathbf{S} \mathbf{S} \mathbf{S}$ and $\mathbf{S} = \mathbf{S} \mathbf{S} \mathbf{S} \mathbf{S}$ and $\mathbf{S} \mathbf{S} \mathbf{S}$

Also, let,

$$
\Gamma = \{ f \in C((0,1], X) : f(0) = x_0 , f(1) = x_1 \}.
$$

 \mathfrak{m} have, Then, we shall have, ll have, μ have, μ *Then, we shall have,*

$$
c = \inf_{f \in \Gamma} \max_{t \in [0,1]} I(f(t)) > -\infty
$$
\n(4.8)

will be a critical value. In other words, $\exists x_0 \in X$ satisfying, $\mathcal{F}_{\mathcal{F}_1}$ \mathcal{L} , \mathcal{L} is in fact above is in fact arc-wise connectedness. Thus, \mathcal{L} is in fact arc-wise connected nested near \mathcal{L}

$$
I(x_0) = c \, , \, I'(x_0) = 0.
$$

nct *arc-wise connectedness* $\frac{1}{2}$ R_{max} $(10, 18, 10)$. The connecting above is in the component above in fact and, M_{max} are M_{max} connecting Remark 4.3.7. The connectedness referred above is in fact arc-wise connectedness. Thus, X \ S $\frac{1}{2}$ (i.e., [10, 1 g 110]). Hence, x_0 and x_1 being in distinct components imprice that, any are in A connecting. **The connectedness referred above is in fact arc-wise connectedness. Thus,** $X \setminus S$ **is indeed a union of open arcw**
constants (one can in X or example, Thus, in and in hyperplane in Tipting and in the boundary of an *X* or **Remark 4.3.7.** The connectedness referred above is in fact arc-wise connectedness. Thus, $X \setminus S$ is indeed a union of open arcwise connected components (ref. [16, Pg-116]). Hence, x_0 and x_1 being in distinct components implies that, any arc in X connecting x_0 and x_1 intercent S intercept *S*.

one can in fact consider X to be a *hyperplane* in X or, the boundary of an open set [in particular, the boundary of being in distinct components implies that, any arc in X connecting x⁰ and x¹ intercept S. For example, one can in fact consider X to be a *hyperplane* in X or, the boundary of an open set [in particular, the boundary of a ball]. \mathcal{F} and \mathcal{F} and \mathcal{F} are $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ are constant Theory Theory $\frac{1}{2}$ and $\frac{1}{2}$ are $\frac{1}{2}$

5 Applications to the Critical Point Theory

Theorem 5.0.1. Suppose, $1 \le q \le p \le \frac{(n+2)}{(n-2)}$, and, $\lambda \in \mathbb{R}$. Also, we consider $\Omega \subset \mathbb{R}^n$ to be a bounded domain. Then $\exists u_0 \in H^1(\Omega)$ $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$ which is in fact a weak solution of the problem, **Theorem 5.0.1.** Suppose, $1 \le q \le p \le$

$$
\begin{cases}\n-\Delta u = |u|^{p-1}u + \lambda |u|^{q-1}u & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega \\
u \neq 0 & \text{in } \Omega\n\end{cases}
$$
\n(5.1)

in the sense that, in the sense that, in the sense that, *in the sense that,*

$$
\int_{\Omega} \nabla u_0 \nabla \phi = \int_{\Omega} |u_0|^{p-1} u_0 \phi + \lambda \int_{\Omega} |u_0|^{q-1} u_0 \phi \qquad \forall \ \phi \in \mathcal{D}(\Omega). \tag{5.2}
$$

fine, $I \in C^1(H^1, (\Omega), \mathbb{I})$ \mathbb{R} *Proof.* We define, $I \in C^1(H^1_0(\Omega), \mathbb{R})$ as,

$$
I(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} - \frac{\lambda}{q+1} \int_{\Omega} |u|^{q+1}.
$$

Clearly, we can verify that, I satisfies (PS). Next we check the conditions for the Mountain Pass Theorem. We obtain, \overline{a} $\frac{1}{1}$ woo $\frac{1}{1}$ $\overline{5}$ applications to the critical point theory of the critical point $\overline{5}$ and $\overline{$

$$
I(0) = 0.
$$

 C can verify that, I satisfies (P S). Next we check the conditions for the \mathcal{C}

 $B) \hookrightarrow L^{p+1}(\Omega)$, $H_0^1(\Omega) \hookrightarrow L^{q+1}(\Omega)$, hence, for some c > 0. Therefore, I(0) = 0. SInce, $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, $H_0^1(\Omega) \hookrightarrow L^{q+1}(\Omega)$, hence, $L^{F^{-1}}(\Omega)$, $H_0(\Omega) \hookrightarrow L^{T^{-1}}(\Omega)$, nence, $H^1(\Omega) \leftrightarrow L^{p+1}(\Omega) \quad H^1(\Omega) \leftrightarrow L^{q+1}(\Omega)$ be $\sum_{i=1}^{n}$

$$
\int_{\Omega} |u|^{p+1} \le c ||u||^{p+1} , \quad \int_{\Omega} |u|^{q+1} \le c ||u||^{q+1}
$$

 $\frac{1}{\sqrt{2}}$ some contract $\frac{1}{\sqrt{2}}$ some contract $\frac{1}{\sqrt{2}}$ for some $c > 0$. Therefore, $\frac{1}{\sqrt{2}}$

$$
I(u) \ge \frac{1}{2}||u||^2 - \frac{c}{p+1}||u||^{p+1} - \frac{c}{q+1}\lambda||u||^{q+1} = \left\{\frac{1}{2}||u|| - c_1||u||^p - c_2\lambda||u||^q\right\}||u||
$$

For $||u|| = R$, we have, \mathbf{S} have, \mathbf{S} are the \mathbf{S} small enough such that, the small enough such that, \mathbf{S} $\mathcal{F}_{\mathcal{A}}^{(1)}$ and $\mathcal{F}_{\mathcal{A}}^{(2)}$ and $\mathcal{F}_{\mathcal{A}}^{(3)}$ and $\mathcal{F}_{\mathcal{A}}^{(4)}$ and $\mathcal{F}_{\mathcal{A}}^{(5)}$ $ve,$

$$
I(u) \ge \left\{ \frac{1}{2}R - c_1 R^p - c_2 R^q \right\} R
$$

Since, $n, q > 1$, thus choosing R small enough such that.

Since, $p, q > 1$, thus choosing R small enough such that, $q₁$, thus choosing α small of

$$
I(u) > a
$$
, on $||u|| = R$ for some $a > 0$.

Let us take any $u_1 \in H_0^1(\Omega)$. Thus, $tu_1 \in H_0^1(\Omega)$ for any $t \in \mathbb{R}$. Now, from the fact that, $I(u) > u$, on $||u|| = R$ for some $u > 0$.
Let us take any $u_1 \in H_0^1(\Omega)$. Thus, $tu_1 \in H_0^1(\Omega)$ for any $t \in \mathbb{R}$. Now, from the fact that, $\frac{1}{2}$

$$
I(tu_1) \longrightarrow -\infty \quad \text{as } t \to \infty.
$$

Choosing *t* large enough such that, $||t_0u_1|| = |t_0| ||u_1|| > R$ and, $I(t_0u_1) < 0$, and, $e = t_0u$, and applying the Mountain Pass Theorem, we can assert that, $\exists u_0 \in H^1(\Omega) \ni I'(u_0) = 0$ and, $I(u_0) \ge a$. assert that, $\exists u_0 \in H^1(\Omega) \ni I'(u_0) = 0$ and, $I(u_0) \ge a$. $a_0 = 11 \, \text{g}$ (as) $\ge 1 \, \text{m}_0$ $=$ and, $\ge 1 \, \text{m}_0$ $=$ as $\alpha_0 \subset H_{0}$ (2z) $\supseteq H_{0}$ and, $\lim_{u \to 0} H_{0}(\alpha_0) = 0$ applying the Mountain Pass Theorem, we can assert that, $\psi =$ $0 = 0$

can also be derived that, $u_0 \neq 0$, as, $a > 0$, and the proof is thus complete.
the other hand *Brezis* and *Niranberg* [28] has developed significant results in observing the model problem: It can also b $\frac{1}{\sqrt{2}}$ It can also be derived that, $u_0 \neq 0$, as, $a > 0$, and the proof is thus complete. I(tu1) −→ −∞ as t → ∞.

It can also be derived that, under that, under the proof is the proof is the proof is thus complete. On the other hand, *Brezis* and *Nirenberg* [28] has developed significant results in observing the model problem: \mathcal{Q} for any t \mathcal{Q} for any the fact that, from the fact that, from the fact that, for any the fact that, \mathcal{Q}

$$
\begin{cases}\n-\Delta u = u^p + \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega \\
u > 0 & \text{in } \Omega\n\end{cases}
$$
\n(5.3)

∪
tant. We can consider the followi an consider the followin $\mathcal{L}_{\mathcal{A}}$ \tan We can consider the followith ↓
tant. We can consider the follow: tant. We can consider the followi $\mathcal{L}_{\mathcal{L}}$ When $p = \frac{(n+2)}{(n-2)}$, $n \ge 3$ and, λ be any real constant. We can consider the following cases. When $p = \frac{(n+2)}{(n-2)}$, $n \ge 3$ and, λ be any real constant. We can consider the following cases. $(n-2)$ $\frac{n+2}{n}$, $n \geq 3$ and, λ be any real constant. We can consider the follow Moreover, the problem does not admit any solution if λ /∈ (0, λ1) and, λ is starshaped.

tant. We can consider the following cases.
(5.3) *indeed has a solution for every* $\lambda \in (0, \lambda_1)$, where, λ_1 denotes the first eig **2.** In case when, $n \ge 4$, the problem (5.3) indeed has a solution for every $\lambda \in (0, \lambda_1)$, where, λ_1 denotes the first eig.
A problem does not admit any solution if $\lambda \notin (0, \lambda_1)$ and λ is starshaned. (5.3) indeed has a solution for ev u = 0 on ∂Ω $\frac{1}{2}$ $\frac{1}{2}$ **Theorem 5.0.2.** In case when, $n \ge 4$, the problem (5.3) indeed has a solution for every $\lambda \in (0, \lambda_1)$, where, λ_1 denotes the first eigenvalue *of* −Δ.

Moreover, the problem does not admit any solution if $\lambda \notin (0, \lambda_1)$ and, λ is starshaped. e problem does not admit any solution if $\lambda \notin (0, \lambda_1)$ and, λ is starshaped. s problem does not admit any solution if $2 \notin (0, 3)$ and 2 is starshaned \mathbf{y} is a solution if \mathbf{y} and \mathbf{y}

modifications being done when n =3, p = 5).

a ball. Subsequently, we shall have that, (5.3) yields a solution iff $\lambda \in (1/4\lambda_1, \lambda_1, \lambda_1)$ being the first eigenvalue of $-\Delta$. **Theorem 5.0.5.** For $n = 5$, the problem (3.5) turns out to be much more detective. In this scenario, a comp Subham De 35 III De 35 III De 35 II De 35 III De 35 II D **Theorem 5.0.3.** *For* $n = 3$, the problem (5.3) turns out to be much more delicate. In this scenario, a complte solution exists only if Ω is , λ 1 being the first eigenvalue of λ .

 $\begin{pmatrix} n+2 \\ n \end{pmatrix}$ in definition $\begin{pmatrix} n+2 \\ n \end{pmatrix}$ **Remark 5.0.4.** In case when $p > \frac{(n+2)}{(n-2)}$, *Brezis* and *Nirenberg* [28] discusses the concept of commenting on the results related to existence of solutions to (5.3) using the notion of general Bifurcation Theory. For example, as mentioned by *Rabinowitz* [29], the problem (5.3) possesses a *component* \mathfrak{C} of solutions (λ, u), which meets (λ , 0) (5.3) possesses a *component* Cof solutions (λ , *u*), which meets (λ ₁, 0) and which is *unbounded* in $\mathbb{R} \times L^{\infty}(\Omega)$.
(5.3) possesses a *component* Cof solutions (λ , *u*), which meets (λ ₁, 0) and whic [−]∆^u ⁼ ^u^p ⁺ λu in Ω *renderg* $[2\delta]$ discusses the concept of commenting on the results re Let $f(x) = \begin{cases} n+2 & n \neq 2, \end{cases}$ by registent of solutions of some of commenting on the results related to $f(x)$

Furthermore, if $p = \frac{(n+2)}{(n-2)}$ and $n \ge 4$, applying the result in Theorem (5.0.2), we can conclude that, the projection of C on the λ -axis does in fact contain the interval $(0, \lambda_1)$ (with appropriate modifications being done when $n = 3$, $p = 5$).

if $p = \frac{(n+2)}{(n+2)}$ and $n \geq 4$, applying the result in Theorem (interval $(0, \lambda_1)$ (with appropriate modifications being done when $n = 3$, $p = 5$). Furthermore, if $p = \frac{(n+2)}{2}$ and $n \ge 4$, applying the result in Theorem (5.0.2), we can conclude that, the projection of C on the λ -axis does in fact contain the interval $(0, \lambda_1)$ (with appropriate modifications being done when $n = 3$, $p = 5$). more, if $p = \frac{(n+2)}{(n-2)}$ and $n \ge 4$, applying the result in Theorem (5.0.2), we can conclude that, the projection of $\frac{1}{\ln 1}$ ati −
− up + kup + k (0.02) , we can conclude ng done when $n = 3$, p Furthermore, if $p = \frac{(n+2)}{(n-2)}$ and $n \ge 4$, applying the result in Theorem (5.0.2), we can conclude that, the projection of C on the λ -axis does in fact contain the interval $(0, \lambda_1)$ (with appropriate modifications being done when $n = 3$, $p = 5$).

 $(n+2)$ As in another scenario when, $p > \frac{(n+2)}{(n-2)}$ and Ω is star-shaped, then the problem (5.3) has no solution for $\lambda \le \lambda^*$, λ^* being some positive constant depending on Ω and p. This was explicitly derived by Rabinowitz [30] for the case when, $n = 3$, $p = 7$. One can in fact use similar argument in the general case by applying Pohozaev's Identity. It another scenario when, $p > \frac{(n-2)}{(n-2)}$ and Ω is star-shaped, then the problem (5.3) has no s

In greater generality as compared to the Dirichlet boundary value problem (5.3), Brezis and Nirenberg [28] also have dealt with the following problem in detail: $\frac{1}{2}$ rester generality as compared to the Dirichlet houndary value problem (5.3). *Brezis* and *Nirenberg* [28] al

$$
\begin{cases}\n-\Delta u = u^p + \lambda u^q & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega \\
u > 0 & \text{in } \Omega\n\end{cases}
$$
\n(5.4)

 $\frac{(n+2)}{(n-2)}$ and, $1 \le q \le p$, $\lambda > 0$ being a constant. Considering different cases for *n*, we can conclude about the existing $\frac{(n+2)}{(n+2)}$ α conclude about the manner as described below. Where, $p = \frac{(n+2)}{(n-2)}$ and, $1 < q < p$, $\lambda > 0$ being a constant. Considering different cases for *n*, we can conclude about the existence of solution (if any) for (5.4) in the manner as described below.

Theorem 5.0.5. *For* $n \geq 4$ *, the problem* (5.4) *indeed admits of a solution for every* $\lambda > 0$ *.*

Theorem 5.0.6. In case when $n = 3$ and consequently, $p = 5$, we can assert the following about existence of solution to the DIrichlet problem (5.4). Theorem 5.0.6. In case when σ and consequently, p σ and σ assert the following about the fol *problem* (5.4): $\mathbf{f}_p = 5$ and can assett the following about existence of solution to the

(i) If $3 \le q \le 5$, \exists solution to the problem for every $\lambda > 0$.
(ii) Equal $\le a \le 3$, solution does gripts only for gufficiently large values of λ (i) If 3 <q< 5, ∃ solution to the problem for every λ > 0. (ii) For $1 \le q \le 3$, solution does exists only for sufficiently large values of λ . \mathcal{L} \mathcal{L}

factor factor \mathcal{C} as \mathcal{C} as a symmetrized as

Brezis Theorem allows us to conclude that, for any function $I \in C^1(X, \mathbb{R})$ which is bounded below, and satisfy (PS) , $\exists u_0 \in X$ satisfying, $I'(u_0) = 0.$ $A \sim \frac{1}{2}$ point for I in case it is bounded below on a finite dimensional subspace of X.

As for another application of Theorem (4.3.1), we next justify the existence of a *critical point* for *I* in case it is bounded below on a finite dimensional subspace of *X*. r application of Theorem (4.3.1), we next justify the existence of a *critical point* for I in case it is bounded below o ubspace of X . T_{S} (Saddle Point Theorem (Rabinowitz) R_{S} to be a Banach α Banach α Banach α

 $V \neq 0$ be a finite dimensional subspace of X and, $X = V \bigoplus E$. Furthermore, we consider that, $\exists R \geq 0$, $\alpha, \beta \in \mathbb{R}$ such that, dimensional subspace of *X.*
Theorem 5.0.7. (Saddle Point Theorem (Rabinowitz) [1]) Assume *X to be a Banach Space and, I* ∈ C¹(X, ℝ) satisfies (PS). Also suppose, $\sum_{i=1}^{n}$ $\sum_{i=1}^{N}$. (5.5). (5.5 *N* (Saddle Point Theorem (Kabinowitz) [1]) Assume X to be a Banach Space and, $I \in C^1(X, \mathbb{R})$ satisfies,
wite dimensional subspace of *X* and, *X* = *V* \bigoplus *E. Eurthermore, we consider that*, $\exists P > 0$, $\alpha, \beta \in \mathbb{R$

$$
\max_{x \in \partial B_K^V(0)} I(x) \le \alpha < \beta \le \inf_{x \in E} I(x). \tag{5.5}
$$

Where,

$$
B_R^V(0) = \{x \in V : ||x|| \le R\} \quad and, \quad \partial B_R^V = \{x \in V : ||x|| = R\}.
$$

Then, $\exists x_0 \in X$ satisfying, $I'(x_0) = 0$. Moreover, the critical value, $c = I(x_0) \ge \beta$, which in fact, can be characterized as,

$$
c = \inf_{S \in \Gamma} \max_{u \in S} I(u). \tag{5.6}
$$

Where,

$$
\Gamma := \left\{ S = \phi(\bar{B}_R^V(0)) \ : \ \phi \in C\left(\bar{B}_R^V(0), X\right) \ and, \ \phi|_{\partial B_R^V} = id \right\}
$$

bove Rabinowitz Theorem requires an application of the *Topological Degree* in \mathbb{R}^n . Proof of the above Rabinowitz Theorem requires an application of the *Topological Degree* in ℝ^{*n*}.

 $\frac{1}{\sqrt{2}}$ and, p $\frac{1}{\sqrt{2}}$, the topological degree (or, Brought), d($\frac{1}{\sqrt{2}}$ **Definition 5.0.1.** (Topological Deg **Definition 5.0.1.** (Topological Degree) Suppose, $\Omega \subset \mathbb{R}^n$ be a bounded and open set. Given $\varphi \in C(\overline{\Omega})$ and, $p \in \mathbb{R}^n \setminus \varphi(\partial \Omega)$, the topological degree (or, *Brouwer Degree*), $d(φ, Ω, p)$ is defined to be an integer satisfying the following properties: _ $\mathbf{5}$ APPLICATIONS TO THE CRITICAL POINT THEORY CRITICAL POINT T

$$
(I) \qquad d(id, \Omega, p) = \begin{cases} 1 & \text{if } p \in \Omega, \\ 0 & \text{if } p \notin \overline{\Omega}. \end{cases}
$$

 $(II) d (\varphi, \varOmega, p) \neq 0 = \Rightarrow \exists q \in \Omega$ $(2) (9)$ d($(9) (9)$ = $(7) 50$ subthat, (9) (II) $d(\varphi, \Omega, p) \neq 0 = \Rightarrow \exists q \in \Omega$ such that, $\varphi(q) = p$.

(III)
$$
d(\varphi, \Omega, p) = 0
$$
 if, $p \notin \varphi(\overline{\Omega})$.

(IV) (Addition-Excision Property) If Ω_1 , $\Omega_2 \subset \Omega$ are open with $\Omega_1 \cap \Omega_2 = \phi$ and Addition-Excision Property) If Ω , $\Omega \subset \Omega$ are open with Ω , $\Omega \subset \phi$ and, $p \notin \mathcal{C}(\overline{\Omega} \setminus (\Omega, \Pi, \Omega))$ then (IV) (Addition-Excision Property) If $\Omega_1, \Omega_2 \subset \Omega$ are open with $\Omega_1 \cap \Omega_2 = \phi$ and, $p \notin \varphi(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$, then, V II) (Product Product Product Product Product Property) is are bounded open sets in $\frac{1}{\sqrt{2}}$ on Property) it s_1 , s_2 \subset sz are open with s_1 \mapsto s_2 = φ and, $p \notin$

$$
d(\varphi,\Omega,p)=d(\varphi,\Omega_1,p)+d(\varphi,\Omega_2,p).
$$

(V) If $\varphi : [0, 1] \times \overline{\Omega} \to \mathbb{R}$ and, $p : [0, 1] \to \mathbb{R}^n$ be continuous, and moreover, $p(t) \notin \varphi(t, \partial \Omega)$ $\forall t \in [0, 1]$, then, $d(\varphi(t, \cdot), \Omega, p(t))$ is independent of *t*. $a(\varphi, s\iota, p) = a(\varphi, s\iota_1, p) + a(\varphi, s\iota_2, p).$
 $1] \times \overline{\Omega} \to \mathbb{R}$ and, $p : [0, 1] \to \mathbb{R}^n$ be continuous, and moreover, $p(t) \notin \varphi(t, \partial \Omega)$ $\forall t \in [0, 1]$, then, $d(\varphi(t, \cdot), \Omega, p(t))$ $R_{\rm eff} = 0.8$ Given $R_{\rm eff} = 0.8$ Given $R_{\rm eff} = 0.6$ Given $\mu_{\rm eff} = 0.00$

 $(VI) d(\varphi_1, \Omega, p) = d(\varphi_2, \Omega, p)$ whenever, $\varphi_1 | \partial \Omega = \varphi_2 | \partial \Omega$. ϕ_1 |*OS2* – ϕ_2 |*OS2*. $m = d(\varphi \Omega n)$ whenever $\varphi |\partial \Omega = \varphi |\partial \Omega$

(VII) (Product Property) If Ω_j 's are bounded open sets in \mathbb{R}^n for every $j = 1, 2$, and, φ_j and p_j are such that, $p_j \in \mathbb{R}^n \setminus \varphi_j(\partial \Omega_j)$, $j = 1, 2$. Then, $R(x_1, y_1, ..., y_n)$ $R(x_2, ..., y_n)$ and one bounded and $p_1(z_1, ..., y_n)$

$$
d(\varphi_1 \times \varphi_2, \Omega_1 \times \Omega_2, (p_1, p_2)) = d(\varphi_1, \Omega_1, p_1) d(\varphi_\mathbb{Q}, \Omega_2, p_2)
$$

Remark 5.0.8. Given Γ and to be bounded and open, φ ∈ C(Γ , R) and, $p \in \mathbb{R}$, φ (Γ), φ can remark the aneally of solutions of the equation, . Given Ω ⊂ Kⁿ to be bounded and open, $\varphi \in C^1(\Omega, \mathbb{R}^n)$ and, $p \in \mathbb{R}^n \setminus \varphi(\alpha \Omega)$, we can relate the theory of the *Brouwe*
ence and multiplicity of solutions of the equation **Remark 5.0.8.** Given $\Omega \subset \mathbb{R}^n$ to be bounded and open, $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ and, $p \in \mathbb{R}^n \setminus \varphi(\partial \Omega)$, we can relate the theory of the *Brouwer Degree* with the existence and multiplicity of solutions of the equation,

$$
\varphi(q) = p \tag{5.7}
$$

 $\mathcal{O}(\mathcal{O}(\log n))$ $\frac{1}{2}$ and $\frac{1}{2}$. In this so called "free" case, the corresponding Drouwer Degree or ψ with respect to sz and p , denoted following expression (q) to be non-singular whenever (5.7) holds true. Then, the *Inverse Function Theorem* yields, (5.7) can have onl σ to be non-given any uponent (5.7) helds two. Then the Inverse Euretien Theorem violds (5.7) can have only number of solutions in Ω . In this so called "nice" case, the corresponding Brouwer Degree of φ with respect to Ω and p , denoted by $d(\varphi, \Omega, p)$ has the following expression, Q , *p*) has the following expression, (q) to be *non-singular* whene Assuming *φ'* (*q*) to be *non-singular* whenever (5.7) holds true. Then, the *Inverse Function Theorem* yields, (5.7) can have only a finite

$$
d(\varphi,\Omega,p)=\sum_{q\in \varphi^{-1}(p)\cap \Omega}sgn\{\det \varphi'(q)\},
$$

where, det A denotes the determinant of a square matrix A .

1. The notion of the *Brouwer Degree* can also be extended from "regular" to "singular" values of C^2 -functions, an inctions on \mathbb{R}^n [ref. [27]]. sgn{det φ′ continuous functions on \mathbb{R}^n [ref. [27]]. **Remark 5.0.9.** The notion of the *Brouwer Degree* can also be extended from "regular" to "singular" values of C²-functions, and then to \ldots in the extended to an infinite dimensional space, in which case it is kown as \ldots

dimensional space, in which case it is kown as Leray-Schauder Degree [ref. [27]]. 10. The definition of *topological degree* as provided above as well as its properties can in fact be extended to an Remark 5.0.10. The definition of *topological degree* as provided above as well as its properties can in fact be extended to an infinite

α rem $(5.0.7)$ $\sum_{i=1}^n$ Proof of Theorem (5.0.7):

 \mathcal{L} . The definition of topological degree as provided above as \mathcal{L} γ $c \in \mathbb{R}$ and $I \in C^1(X, \mathbb{R})$, we define the following sets, *Proof.* For any $c \in \mathbb{R}$ and $I \in C^1(X, \mathbb{R})$, we define the following sets,

$$
K := \big\{ x \in X \ : \ I'(x) = 0 \big\}
$$

and,

$$
K_c := \{ x \in K \; : \; I(x) = c \} \, .
$$

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 28 $\frac{P}{R}$ For any c $\frac{P}{R}$ and $\frac{P}{R}$, we define the following sets, we define the following se

First, we intend to prove that, $K_c \neq \phi$ for c as mentioned in (5.6). Assume if possible that, $K_c = \phi$ in this case. We choose ϵ as, $\begin{array}{ccc} 1 & \rightarrow & c & t \\ & & \end{array}$ nd to prove that, $K_c \neq \phi$ for c as mentioned in (5.6). Assume if possible that, $K_c = \phi$ in this case. We choose ϵ as, First, we intend to prove that, $K_c \neq \phi$ for c as mentioned in (5.6). Assume if possible that, $K_c = \phi$ in this case. We choose ϵ as, $\mathcal{L}(\mathcal{$ 5.6). Assume if possible that, $K_c = \phi$ in this case. We choose ϵ as, First, we intend to prove that, $K_c \neq \phi$ for c as mentioned in (5.6). Assume if possible that, $K_c = \phi$ in this case. We choose ϵ as, $K_{\rm eff}$ in this case. We choose $K_{\rm eff}$ as a symmetric case. We choose $K_{\rm eff}$

 $\overline{1}$

$$
0 < \epsilon < \frac{1}{4}(\beta - \alpha) \tag{5.8}
$$

 $\beta \subset \Gamma$ such that, and, consider $S \in \Gamma$ such that, $\mathcal{A} = \mathcal{A} \cup \mathcal{A}$ such that, consider $\mathcal{A} = \mathcal{A} \cup \mathcal{A}$ $S \in \Gamma$ such that,
 $S \in \Gamma$ such that, $\mathcal{L} = \mathcal{L}$

 $\mathcal{F}_{\mathcal{F}}$ for case that, $\mathcal{F}_{\mathcal{F}}$ for case mentioned in (5.6). Assume if p

and, consider
$$
S \in \Gamma
$$
 such that,

$$
\max_{x \in S} I(x) < c + \epsilon. \tag{5.9}
$$

x∈S [25, Pg. 27]) satisfying the following conditions, • $\eta(t, x) = x$ if, $|I(x) - c| \ge 2\epsilon$. S)_c condition is satisfied under these assumptions. Let, $\eta : [0, 1] \times X \rightarrow X$ be a I-decreasing homotopy (ref. Coroll $\lim_{x \to a}$ $\lim_{x \to a}$ Such that, $(PS)_c$ condition is satisfied under these assumptions. Let, $\eta : [0, 1] \times X \to X$ be a I-decreasing homotopy (ref. Corollary (1.7) Such that, $(PS)_c$ condition is satisfied under these assumptions. Let, $\eta : [0, 1] \times X \to X$ be a I-decreasing homotopy (ref. Corollary (1.7)

- $\begin{array}{ccc} \n\hline\n\end{array}$ (ref. Corollary $\begin{array}{ccc} \n\hline\n\end{array}$) satisfying the following conditions, $\hline\n\end{array}$ $\mathcal{L}_{\eta}(v, w) = w \ln, \|\mathcal{L}(w) - v\| \leq 2c.$ $|c| \geq 2\epsilon$. $\sum_{i=1}^{n}$
- $f^{c-\epsilon}$, where, $I^c := \{x \in X : I(x) \}$ $f^{c-\epsilon}$, where, $I^c := \{x \in X : I(x) \leq c\}.$ $Q(1, I^{c+\epsilon}) \subset I^{c-\epsilon}$, where, $I^c := \{x \in X : I(x) \leq c\}.$ • $\eta(1, I^{c+\epsilon}) \subset I^{c-\epsilon}$, where, $I^c := \{x \in X : I(x) \le c\}.$ \bullet $\eta(1,1) \subset I$, where, $1 \cdot -\{1 \in A \cdot I(1)\leq 7\}$ • $\eta(1, I^{c+\epsilon}) \subset I^{c-\epsilon}$, where, $I^c := \{x \in X : I(x) \le c\}.$ \mathbf{r} as \mathbf{r} $\$ • $\eta(1, I^{c+\epsilon}) \subset I^{c-\epsilon}$, where, $I^c := \{x \in X : I(x) \le c\}.$

 $(1, S)$. If $S_1 \in \Gamma$, then by (5.9) we derive, = $\eta(1, S)$. If $S_1 \in \Gamma$, then by (5.9) we derive, denote, $S = n(1, S)$. If $S \in \Gamma$ then by (5.9) we derive denote, $S_1 = \eta(1, S)$. If $S_1 \in I$, then by (5.9) we derive, We denote, $S_1 = \eta(1, S)$. If $S_1 \in \Gamma$, then by (5.9) we derive, $\frac{1}{1}$ \cdots , $\frac{1}{1}$ \cdots

 $\mathcal{L} \left(\mathcal{L} \right)$ $\mathcal{L} \left(\mathcal{L} \right)$

 $\frac{1}{2}$ $\frac{1}{2}$

$$
\max_{x \in S_1} I(x) \le c - \epsilon
$$

ficts the definition of c . which contradicts the definition of c .

x
≈S ϵ $C(\bar{D}V(0), V)$ and $C = \pm (\bar{D}V(0))$ of \sum_{R} (0), X) and, \sum_{I} φ_{I} (φ_{R} (0)). Thence, \sum_{I} C \sum_{I} \sum_{I} () blish that, $S_1 \in \Gamma$. Consider $\phi \in C(\bar{B}_R^V(0), X)$ such that, $\phi|_{\partial B_R^V(0)} = id$ and, $= \phi(\bar{B}_R^V(0))$. We thus $\mathcal{L} \in C(\bar{B}_R^V(0), X)$ and, $S_1 = \phi_1(\bar{B}_R^V(0))$. Hence, $S_1 \in \Gamma \phi_1(x) = x, \forall x \in \partial B_R^V(0)$. $\big)$). Let us establish that, $S = \sum_{i=1}^{n} C_i = \frac{\sum_{i=1}^{n} S_i}{N}$ blish that, $S_1 \in \Gamma$. Consider $\phi \in C(B_R^{\nu}(0), X)$ such that, $\phi|_{\partial B_R^{\nu}(0)} = id$ and, $\phi(B_R^{\nu}(0))$. We thus $\in C(B_R^V(0), X)$ and, $S_1 = \phi_1(B_R^V)$ Let us establish that, $S_1 \in \Gamma$. Consider $\phi \in C(B_R^{\nu}(0), X)$ such that, ϕ $\partial_{\beta} B_R^V(0) = i a$ and $\phi_1 = \eta(1, \phi) \in C(\bar{B}_R^V(0), X)$ and, $S_1 = \phi_1(\bar{B}_R^V(0))$. Hence, $S_1 \in \Gamma \phi_1(x) = x, \quad \forall x \in \partial B_R^V$ $\text{Consider } \phi \in C$ et us establish that, $S_1 \in \Gamma$. Consider $\phi \in C(\bar{B}_R^V(0), X)$ such that, $\phi|_{\partial B_R^V(0)} = id$ and, $=\phi(\bar{B}_R^V(0))$. We thus have, $\mathbf{L}^{(0)}$ X) and, $S_1 = \phi_1(\bar{B}_R^V(0))$. Hence, $S_1 \in \Gamma \phi_1(x) = x$, $\forall x \in \partial B_R^V(0)$. Let us establish that, $S_1 \in \Gamma$. Consider $\phi \in C(\overline{B}_R^V(0), X)$ such that, $\phi|_{\partial B_R^V(0)} = id$ and, $= \phi(\overline{B}_R^V(0))$. We thus have, dina, $S_1 = \varphi_1(\mathcal{D}_R(\mathbf{0}))$. Hence, $S_1 \subset \Gamma$, $\Gamma(\mathbf{0})$, $\Gamma(\mathbf{0})$, $\Gamma(\mathbf{0})$, $\Gamma(\mathbf{0})$ $\mathcal{L} \in C(\bar{B}_{\scriptscriptstyle B}^V(0),X)$ and, $S_1 = \phi_1(\bar{B}_{\scriptscriptstyle B}^V(0))$. Hence, $S_1 \in \Gamma \phi_1(x) = x$, $\forall x \in \partial B_{\scriptscriptstyle B}^V(0)$ Θ). $\phi_1 = \eta(1, \phi) \in C(\bar{B}_R^V(0), X)$ and, $S_1 = \phi_1(\bar{B}_R^V(0))$. Hence, $S_1 \in \Gamma \phi_1(x) = x$, $\forall x \in \partial B_R^V(0)$. $\big)$). S το του του προσπαθεί του θα προ $\varphi_1 = \eta(1, \varphi) \in C(D_R(0), A)$

 $\mathfrak{t},$ $\mathfrak{t}, \mathfrak{t}$ $\frac{1}{2}$ if, $\frac{1}{2}$ We claim that, W_{θ} claim that W_C channel that $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak{a}_6, \mathfrak{a}_7, \mathfrak{a}_8, \mathfrak{a}_9, \mathfrak{a}_9, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak{a}_7, \mathfrak{a}_8, \mathfrak{a}_9, \mathfrak{a}_9, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak$

$$
c \ge \beta. \tag{5.10}
$$

 ϵ β. (5.10) $(5.8),$ S hold true, then, for $x \in \partial B_R^1(0)$, using (5.5) and (5.8), If (5.10) does hold true, then, for $x \in \partial B_R^V(0)$, using (5.5) and (5.8), $R_{\rm g}$ (5.6), If (5.10) does hold true, then, for $x \in \partial B_N^V(0)$, using (5.5) and (5.8),

$$
I(x) \le \alpha < \alpha + 2\epsilon < \beta - 2\epsilon \le c - 2\epsilon.
$$

s assert that, $\phi_1(x) = x$. T_{max} $\frac{1}{\sqrt{1}}$ and P₁ (w). We This helps us assert that, $\phi_1(x) = x$. \sum_{x} \sum_{x} \sum_{y} \sum_{y} \sum_{z} \sum_{z This helps us assert that, $\phi_1(x) = x$. T is only sufficiently sufficiently sufficient \mathcal{L} and \mathcal{L} on \mathcal{L} on \mathcal{L} onto \mathcal{L} on \mathcal{L} on

only suffices to show (5.10). Wednote P_1 a th \mathbb{R}^n for some n. For $\phi \in C(\bar{B}_P^V(0), X)$, $\phi = id$ on $\partial B_P^V(0)$. Hence, by properties (I) and (VI) in the d $(5.0.1)$ of the $\emph{Topological Degree},$ we obtain, T_{min} measure that, $\frac{1}{2}$ as the set of the $\frac{1}{2}$. $\frac{1}{\sqrt{2}}$ is only sufficiently sufficiently sufficiently for an $\frac{1}{2}$ and $\frac{1}{2}$ on $\frac{1}{2}$ V with \mathbb{R}^n for some n. For $\phi \in C(\bar{B}_R^V(0), X)$, $\phi = id$ on $\partial B_R^V(0)$. Hence, by properties (I) and (VI) in the definition only suffices to show (5.10). We dnote P_1 and P_2 to be the projections of X onto V and E respectively. Furth Therefore, it only suffices to show (5.10). We dnote P_1 and P_2 to be the projections of X onto V and E respectively. Furth \mathbb{F} id \mathbb{F}^n for \mathbb{F} and \mathbb{F}^N (\mathbb{F}^N) of \mathbb{F}^N or \mathbb{F}^N or W Therefore, it only suffices to show (5.10). We
dnote P_1 and P_2 to be the projections of X onto
V and E respectively. Furthermore, we identify V with \mathbb{R}^n for some n For $\phi \in C(\overline{B}_K^V(0), X)$ $\phi = id$ on $\partial B_K^V(0)$. Hence, by proper Degree, we obtain, we identify V with \mathbb{P}^n for some n. For $\phi \in C(\bar{R}^V(\mathfrak{O}) \mid X)$, $\phi = id$ on $\partial B^V_{\mathfrak{O}}(\mathfrak{O})$. Hence by pro We identify V with as for some *R*. For $\varphi \in \mathcal{O}(D_R^1(\sigma), \Lambda)$, τ and $\varphi \in \mathcal{O}(D_R^2(\sigma), \Lambda)$ $\text{vertices } (I) \text{ and } (VI) \text{ in the definition}$ we identify V with \mathbb{R}^n for some n. For $\phi \in C(\bar{B}_R^V(0), X)$, $\phi = id$ on $\partial B_R^V(0)$. Hence, by properties (I) and (VI) in the definition T_{max} is only sufficient to show ($\frac{1}{2}$ and $\frac{1}{2}$ and P2 to be the projections of $\frac{1}{2}$ on $\frac{1}{2}$ onto only suffices to show (5.10). We
dnote P_1 and P_2 to be the projections of X onto
V and E respectively. Furthermore, we identify V with \mathbb{R}^n for $V = \frac{\alpha}{\sqrt{2}} V_{\text{S}}(z)$, $\mathbf{v} = \frac{1}{2} \mathbf{v} \mathbf{v}$ V with \mathbb{R}^n for some *n*. For $\phi \in C(\overline{B}_R^V(0), X)$, $\phi = id$ on $\partial B_R^V(0)$. Hence, by properties (*I*) and (V*I*) in the definition (5.0.1) of the Topological Degree, we obtain, T is only sufficiently sufficiently sufficiently sufficiently sufficiently sufficiently of \mathbf{X}^T on \mathbf{X}^T V and E respective values to show (5.10). We did F_1 and F_2 to be the projections of λ onto V and E respective $\frac{R}{L}$ \forall with \mathbb{R}^n for some *n*. For $\phi \in C(\overline{B_R^V}(0), X)$, $\phi = id$ on $\partial B_R^V(0)$. Hence, by properties (*I*) and (*VI*) in the d Therefore, it only suffices to show (5.10). We
dnote P_1 and P_2 to be the projections of X onto
 V and E respectively. Furthermore, $\Sigma_{\rm tot}$ on $\Sigma_{\rm tot}$ and $\Sigma_{\rm tot}$ and $\Sigma_{\rm tot}$ and $\Sigma_{\rm tot}$ in the $\Omega_{\rm tot}$ in the Topological (5.0.1) or the $\Omega_{\rm tot}$ \mathbf{w} be interesting \mathbf{v} we regularity V with \mathbb{R}^n for some r , $\text{For } \phi \in C(\bar{D}V(\mathfrak{O}) \mid X)$, $\phi = id$ on $\partial R^V(\mathfrak{O})$. Hence, by properties (I) and (VI) in Therefore, it only suffices to show (5.10). We
different P_1 and P_2 to be the projections of Λ onto
 V and we identify V w $\frac{1}{2}$ are concerning to the critical point $\frac{1}{2}$

$$
d(P_1\phi, B_R^V(0), 0) = d(id, B_R^V(0), 0) = 1.
$$

 $x_0 \in B_R^1(0)$ satisfying, $P_1\phi(x_0) = 0$. Conseq Applying property (*II*) in definition (5.0.1), $\exists x_0 \in B_R^V(0)$ satisfying, $P_1\phi(x_0) = 0$. Consequently, for each $S = \phi(\bar{B}_R^V(0)) \in \Gamma$, $\exists x_0 \in B_R^V(0)$ such that, perty (II) in definition $(5.0.1)$, $\exists x_0 \in B_R^V(0)$ satisfying, $P_1\phi$

$$
\phi(x_0) = P_2 \phi(x_0) \in E.
$$

On the other hand, from (5.5) , we can conclude,

$$
\max_{x \in \bar{B}_R^V(0)} I(\phi(x)) \ge I(\phi(x_0)) \ge \beta.
$$

Curr Res Stat Math, 2024 Volume 3 | Issue 3 | 29 $U_{\rm{SUSY}}$ (5.6), it follows that, c $B_{\rm{SUSY}}$ is the proof is thus complete. $\mathbf{Math, } 2024$

Using (5.6), it follows that, $c \ge \beta$, and the proof is thus complete. ^x∈B¯^V $U(\mathbf{S}|\mathbf{S})$, it follows that, c \mathbf{S} over all surfaces modelled on BV surfaces modelled on It follows that, $c \geq \beta$, and the proof is \mathbf{R} and sumplemed the same boundary. Unlike the Mountain-Passeus the Mountain-Passeus the Mountain-

^R (0)

Remark 5.0.11. Heuristically speaking, in the above Theorem (5.0.7), c is the minimax of I over all surfaces modelled on $B_R^V(0)$, sharing the same boundary. Unlike the *Mountain-Pass Theorem*, in applications of the *Saddle Point Theorem*, in general, no critical points of I are known initially. Important to note that, the condition (5.5) are satisfied if I is *convex* on E , $\textit{concave}$ on V , and appropriately *coercive*. concave on V , and appropriately coercive. \therefore concave on V , and appropriately *coercive*. α appropriately correct. And appropriately coordinate. known increases modelled on $B_R^V(0)$, sharing the same boundary. Unlike the *Mountain-Pass*

 $\frac{1}{\sqrt{2}}$ if $\frac{1}{\sqrt{2}}$ if $\frac{1}{\sqrt{2}}$ if $\frac{1}{\sqrt{2}}$

 $R_{\rm eff}$, in the above Theorem (5.0.11. Here, in the above Theorem (5.0.7), c is the minimax of Indian α

Another version of the *Rabinowitz Saddle Point Theorem* (Theorem $(5.0.7)$) can be found in [26]. $A = \frac{1}{2}$ [26]. 26 .

Theorem 5.0.12. Given a real Banach Space X having the following direct sum decomposition, $X = V \oplus E$, where V and E are closed subspaces with $\dim V < \infty$. Suppose, $I \in C^1(X, \mathbb{R})$ $satisfy (PS)$ condition, and, satisfy (P S) condition, and, where V and E are closed subset

$$
I(x) \longrightarrow -\infty \text{ as } ||x|| \to \infty \text{ for } x \in X \tag{5.11}
$$

and, and, *and,*

$$
\inf_{y \in E} I(x) = d > -\infty
$$
\n(5.12)

 $Defne, D := \{x \in V : ||x|| \leq D\}$ with P is chosen so large that, $x \in V$ and $||x|| = P \rightarrow$ Define, $D := \{x \in V : ||x|| \le R\}$ with R is chosen so large that, $x \in V$ and $||x|| = R \Longrightarrow$ $I(x) < d.$ If, \overline{B} -1 $\mu \in V$

$$
\Gamma := \{ \phi \in C(D, X) \; : \; \phi|_{\partial D} = id \}
$$
\n
$$
(5.13)
$$

Then, *Then*

I(x) < d.

$$
c = \inf_{\phi \in \Gamma} \max_{x \in D} I(\phi(x)) > -\infty
$$
\n(5.14)

and, $\exists u_0 \in X$ satisfying, $I(u_0) = c$ and, $I'(u_0) = 0$.

Remark 5.0.13. For other versions of proof of Theorem (5.0.7) and, other important applications of the Mountain Pass Theorem, one can Subham De 40 III De Subham De 40 III De
Liste de 40 II De 40 III De 40 refer to [1-25].

Remark 5.0.14. Theorem (5.0.7) essentially states that under certain conditions, a functional (a function of functions) will have at least one critical point that is a saddle point. This critical point is where the functional doesn't increase or decrease, representing a sort of equilibrium.

To put it in a more formal geometric context, consider a functional defined on an infinitedimensional space. The space can be thought of as a "landscape" of all possible functions. The functional assigns a "height" (or value) to each function in this landscape. Rabinowitz's theorem guarantees that there's at least one function in this landscape that has a saddle point: it's not the highest or lowest point, but it's a point of balance between different "directions" in the function space. This geometric interpretation is quite abstract because we're dealing with spaces that are not easy to visualize. However, the concept of a saddle point as a point of equilibrium remains a powerful image to understand the essence of the theorem.

Statements and Declarations Conflicts of Interest Statement

I as the author of this article declare no conflicts of interest.

Data Availability Statement

I as the sole author of this article confirm that the data supporting the findings of this study are available within the article [and/or] its supplementary materials.

References

- 1. Rabinowitz, P. H. (Ed.). (1986). *Minimax methods in critical point theory with applications to differential equations* (No. 65). American Mathematical Soc..
- 2. [Ambrosetti, A., & Rabinowitz, P. H. \(1973\). Dual variational methods in critical point theory and applications.](https://doi.org/10.1016/0022-1236(73)90051-7) *Journal of functional Analysis, 14*[\(4\), 349-381.](https://doi.org/10.1016/0022-1236(73)90051-7)
- 3. Bahuguna, D., Raghavendra, V., & Kumar, B. R. (Eds.). (2002). *Topics in sobolev spaces and applications*. Alpha Science Int'l Ltd..
- 4. Stein, E. M., & Shakarchi, R. (2011). *Fourier analysis: an introduction* (Vol. 1). Princeton University Press.
- 5. Hörmander, L. (1963). Linear partial differential operators Springer-Verlag. *New York.*
- 6. Adams, R. A., & Fournier, J. F. (1975). Sobolev spaces, acad. *Press, New York, 19*(5).
- 7. Evans, L. C. (2022). *Partial differential equations* (Vol. 19). American Mathematical Society.
- 8. V. G. Mazya. (1985). *Sobolev Spaces*, Springer-Verlag, Springer Series in Soviet Mathematics.
- 9. Sobolev, S. L. (1936). On some estimates relating to families of functions having derivatives that are square integrable. In *Dokl. Akad. Nauk SSSR* (Vol. 1, No. 10, pp. 267-270).
- 10. [Sobolev, S. L. \(1938\). On a theorem of functional analysis.](https://cir.nii.ac.jp/crid/1571135649719480320) *Mat. Sbornik, 4,* 471-497.
- 11. 山田義雄[. \(1994\). Michael Struwe, Variational Methods; Applications to Nonlinear Partial Differential Equations and Hamiltonian](https://doi.org/10.11540/bjsiam.4.1_93) [Systems, Springer-Verlag, 1990, XIV+ 244pp.](https://doi.org/10.11540/bjsiam.4.1_93) 応用数理*, 4*(1), 93-94.
- 12. Ziemer, W. P. (2012). *[Weakly differentiable functions: Sobolev spaces and functions of bounded variation](https://sites.math.rutgers.edu/~brezis/PUBlications/123-journal.pdf)* (Vol. 120). Springer Science [& Business Media.](https://sites.math.rutgers.edu/~brezis/PUBlications/123-journal.pdf)
- 13. Brezis, H., & Nirenberg, L. (1991). Remarks on finding critical points. *Communications on Pure and Applied Mathematics, 44*(8‐9), 939-963.
- 14. Shuzhong, S. (1996). Convex Analysis and Nonsmooth Analysis. *ICTP Notes.*
- 15. De Figueiredo, D. G. (1989). *[Lectures on the Ekeland variational principle with applications and detours](https://mathweb.tifr.res.in/sites/default/files/publications/ln/tifr81.pdf)* (Vol. 81). Berlin: Springer.
- 16. J. Dungundji (1966). *Topology*, Allyn & Bacon.
- 17. Zou, W., & Schechter, M. (2006). *Critical point theory and its applications.* Springer Science & Business Media.
- 18. Daniela Kraus, Oliver Roth, *[Critical points of inner functions, nonlinear partial differential equations, and an extension of Liouville's](https://doi.org/10.1112/jlms/jdm095) theorem*[, Journal of the London Mathematical Society 77.1 \(2008\): 183-202.](https://doi.org/10.1112/jlms/jdm095)
- 19. Mawhin, J. (2013). *Critical point theory and Hamiltonian systems* (Vol. 74). Springer Science & Business Media.
- 20. [Bahri, A., & Berestycki, H. \(1981\). A perturbation method in critical point theory and applications.](https://www.ams.org/journals/tran/1981-267-01/S0002-9947-1981-0621969-9/) *Transactions of the American [Mathematical Society, 267](https://www.ams.org/journals/tran/1981-267-01/S0002-9947-1981-0621969-9/)*(1), 1-32.
- 21. [Bahri, A., & Berestycki, H. \(1981\). A perturbation method in critical point theory and applications.](https://www.ams.org/journals/tran/1981-267-01/S0002-9947-1981-0621969-9/) *Transactions of the American [Mathematical Society, 267](https://www.ams.org/journals/tran/1981-267-01/S0002-9947-1981-0621969-9/)*(1), 1-32.
- 22. Ghoussoub, N. (1993). *Duality and perturbation methods in critical point theory* (No. 107). Cambridge University Press.
- 23. Palais, R. S., & Terng, C. L. (2006). *Critical point theory and submanifold geometry* (Vol. 1353). Springer.
- 24. Kreyszig, E. (1991). *Introductory functional analysis with applications* (Vol. 17). John Wiley & Sons.
- 25. do Rosário Grossinho, M., & Tersian, S. A. (2013). *An introduction to minimax theorems and their applications to differential equations* (Vol. 52). Springer Science & Business Media.
- 26. Rabinowitz, P. H. (1991). A note on a semilinear elliptic equation on Rn. *Nonlinear Analysis: A Tribute in Honour of G. Prodi, Quad. Scu. Norm. Super. Pisa*, 307-318.
- 27. Deimling, K. (2013). *Nonlinear functional analysis*. Springer Science & Business Media.
- 28. [Brézis, H., & Nirenberg, L. \(1983\). Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents.](https://sites.math.rutgers.edu/~brezis/PUBlications/98-journal.pdf) *Communi[cations on pure and applied mathematics, 36](https://sites.math.rutgers.edu/~brezis/PUBlications/98-journal.pdf)*(4), 437-477.
- 29. [Rabinowitz, P. H. \(1971\). Some global results for nonlinear eigenvalue problems.](https://doi.org/10.1016/0022-1236(71)90030-9) *Journal of functional analysis, 7*(3), 487-513.
- 30. [Rabinowitz, P. H., & Moser, J. K. \(1974\). Variational methods for nonlinear elliptic eigenvalue problems.](https://www.jstor.org/stable/24890752) *Indiana University Mathe[matics Journal, 23](https://www.jstor.org/stable/24890752)*(8), 729-754.

Copyright: ©2024 Subham De. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.