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An Exposition on Critical Point Theory with Applications

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Abstract

Critical Point Theory plays a pivotal role in the study of Partial Differential Equations (PDEs), particularly in investigating the existence, uniqueness, and multiplicity of weak solutions to elliptic PDEs under specified boundary conditions. This article offers a concise survey of key concepts, including differentiation on Banach spaces, the analysis of maxima and minima, and their applications to PDEs. Additionally, we explore the construction of weak topologies on Banach spaces before introducing the Variational Principle and its relevance to solving PDEs. In the final section, we focus on results pertaining to the existence of weak solutions for Dirichlet boundary value problems under specific conditions. A highlight of this work is the proof of Rabinowitz's Saddle Point Theorem via the Brouwer Degree method. Researchers interested in exploring the themes covered in this paper will find the reference section to be a valuable resource for further study.

Keywords: Critical Point, Saddle Point, Banach Spaces, Weak Convergence, Variational Principle, Weak Solution, Mountain Pass Theorem, Rabinowitz Saddle Point Theorem, Brouwer Degree

1 Introduction

1.1 Preliminaries

A priori given a *partial differential* equation on a bounded domain, be it *linear* or *non-linear*, we might end up obtaining solutions which also happens to be *critica points* of certain functionals defined on some appropriate *Sobolev Space*. Suppose, we consider the following boundary value problem on some bounded domain $\Omega \subset \mathbb{R}^n$ as follows:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

Where, $\partial\Omega$ denotes the *boundary* of Ω in $\mathbb{R}^n$. A priori from the boundary condition, it only suffices to search for weak solutions of (1.1) in the Sobolev Space $H_0^1(\Omega)$, the later assertion follows from the fact that, not all functions in $H_0^1(\Omega)$ is smooth (i.e. C^2).

Definition 1.1.1. (Weak Solution) A function $u \in H_0^1(\Omega)$ is defined to be a weak solution of (1.1) if,

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega f\phi \qquad \forall\ \phi \in H_0^1(\Omega) \tag{1.2}$$

In case when u is a *classical solution*, i.e., in other words, when, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and satisfies (1.1) point-wise, then, we can indeed obtain (1.2) by multiplying (1.1) by ϕ and integrating by parts.

***Remark* 1.1.1**. (1.2) is valid only when, $f \in H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)^*$.

Define a functional, $J:\ H_0^1(\Omega) \longrightarrow \mathbb{R}$ as,

$$J(u) := \frac{1}{2}\int_{\Omega} |\nabla u|^2 - \int_{\Omega} fu \tag{1.3}$$

It can be deduced that, J is in fact a C^1-function, and, $J'(u) : H_0^1(\Omega) \rightarrow \mathbb{R}$ is a bounded linear functional having the following expression,

$$J'(u)\phi = \int_{\Omega} \nabla u \nabla \phi - \int_{\Omega} f\phi \qquad \forall\ \phi \in H_0^1(\Omega) \tag{1.4}$$

Hence, we can infer that, $u \in H_0^1(\Omega)$ is a weak solution of (1.1) $\Leftrightarrow J'(u_0) = 0$.

1.2 Conditions on a C^1-Function Defined on a Real Banach Space

Given a real *Banach Space* X and a C^1-function, $I : X \rightarrow \mathbb{R}$, our primary objective in this section shall be to obtain certain conditions on I in order to ensure that, $\exists\ x_0 \in X$ satisfying, $I'(x_0) = 0$.

The answer is quite simple for the one-dimensional case. Consider the example when, $X = \mathbb{R}$. Correspondingly, choose any C^1-function, $I : \mathbb{R} \rightarrow \mathbb{R}$. Then, for every $x_1, x_2 \in \mathbb{R}$ with, $x_1 < x_2$, if $\exists\ x_3 \in (x_1, x_2)$ satisfying, $I(x_3) > max\{I(x_1), I(x_2)\}$, it follows that, $\exists\ x_0 \in (x_1, x_2)$ such that, $I'(x_0) = 0$.

Although, for higher dimensional cases, it's rather difficult. For example, if we consider, $I : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as, $I(x, y) := e^x - y^2$. We can deduce that, for any (x_1, y_1) and $(x_1, -y_1)$ in $\mathbb{R}^2$ with $x_1, y_1 > 0$, $I(x_1, \pm y_1) < 0$ and, $I(x, 0) = e^x > 0$. This implies that, the line, $y = 0$ separates the points $(x_1, \pm y_1)$ where,

$$I(x,0) > max\{I(x_1, y_1), I(x_1, -y_1)\}, \qquad \forall\ x \in \mathbb{R}$$

But, an important observation that, $I'(x, y) = (e^x, -2y)$ helps us conclude that, $\nexists$ any critical *point* for I.

1.3 Palais-Smale Condition

Definition 1.3.1. (Palais-Smale Condition) Given a Banach Space X and, $I \in C^1(X;\mathbb{R})$, we define the functional I to satisfy the Palais-Smale Condition, if every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which $I(x_n)$ is *bounded* and $I'(x_n) \rightarrow 0$ contains a convergent subsequence.

We can infer about the above problem as discussed in the section (1.2) in case for an arbitrary Banach Spaces using a famous result by *Ambrosetti* and *Rabinowitz*.

Theorem 1.3.1. (*Mountain-Pass Theorem*) *Given a Banach Space X and, $I \in C^1(X;\mathbb{R})$, assume that, I satisfies the Palais-Smale Condition. Furthermore, suppose, $\exists\ R > 0$ and, $e \in X$ satisfying, $\|e\| > R$ and, $b = \inf_{x\in\partial B_R(0)} I(x) > max\{I(0), I(e)\}$. Then, $\exists\ x_0 \in X$ such that, $I'(x_0) = 0$ and, $I(x_0) \geq b$.*

Remark **1.3.2.** In other words, the theorem implies that, if a pair of points in the graph of I are indeed separated by a mountain $\partial B_R(0)$, then I has a *critical point*.

Remark **1.3.3**. One of our aims in this article is to present aproof of the *Mountain-Pass Theorem* (1.3.1), as well as applying the statement of the same in order to look for non-negative weak solution $u \in H^1{}_0(\Omega)$ to the non-linear boundary value problem (1.1) over a bounded domain $\Omega \subset \mathbb{R}^n$, $f : \mathbb{R} \rightarrow \mathbb{R}$ being continuous.

1.4 Differentiation in Banach Spaces

Suppose, we choose any two real Banach Spaces X and Y , and, $B(X; Y)$ denotes the space of all bounded linear operators from X to Y. Let us further denote, $X^* := B(X;\mathbb{R})$. Moreover, let $A \subset X$ be *open*.

Definition 1.4.1. Let, $x_0 \in A$. A function, $I : A \rightarrow Y$ is defined to be *Frechet Differentiable* (or, just *differentiable*) at x_0, if $\exists$ a bounded linear operator, $DI(x_0) : X \rightarrow Y$, in other words, $DI(x_0) \in B(X; Y)$ satisfying,

$$\lim_{h\to 0} \frac{||I(x_0+h) - I(x_0) - DI(x_0)h||_Y}{||h||_X} = 0 \tag{1.5}$$

***Example* 1.4.1.** For any differentiable function, $f : \mathbb{R} \to \mathbb{R}$ at some $x_0 \in \mathbb{R}$, we have, $f'(x_0) \in \mathbb{R}$. Correspondingly, the Frechet Derivative, $Df(x_0)$ of f at the point x_0 has the following expression, $Df(x_0)(x) = f'(x_0).x$, $\forall\, x \in \mathbb{R}$.

In general, in case when, $X = \mathbb{R}$, $A = (a, b)$ and, $I : A \to Y$ be *differentiable* at $x_0 \in (a, b)$, then we define, $I'(x_0) := DI(x_0)(1) \in Y$ via the canonical isomorphism, $i : B(\mathbb{R}; Y) \to Y$ given by, $i(T) := T(1)$.

For arbitrary real Banach Spaces X and Y, if I : $A \to Y$ be Frechet Differentiable at some $x_0 \in A$, then, corresponding to the *Frechet Derivative* of I at x_0 defined as, $DI(x_0)$ satisfying (1.5), we write $I'(x_0)$ to be the derivative of I at x_0.

Definition 1.4.2. A function, $I : A \to Y$ is defined to be differentiable on A if it is differentiable at every point of A. Furthermore, $I \in$ C1(A; Y) if, $I' : A \to B(X; Y)$ is continuous.

Example 1.4.2. (*i*) Suppose, $(X, \langle ., . \rangle)$ be a *Hilbert Space* and, $I : X \to \mathbb{R}$ defined as, $I(x) = \langle x, x \rangle$. Then, $I \in C^1(X;\mathbb{R})$ such that,

$$I'(x)y = 2\langle x, y \rangle, \qquad x, y \in X.$$

(*ii*) Suppose, $f \in H^{-1}(\Omega)$. Define, $J : H^1_0(\Omega) \to \mathbb{R}$ as,

$$j(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu.$$

We can in fact conclude that, $J \in C^1(H^1_0(\Omega);\mathbb{R})$. Furthermore,

$$J'(u)\phi = \int_\Omega \nabla u \nabla \phi - \int_\Omega f\phi \qquad \forall\ u, \phi \in H^1_0(\Omega).$$

We can comment on some important properties of the derivative as follows.

1. (Chain Rule) A priori given X, Y and Z, and, $A \subset X$, $B \subset Y$ being non-empty open sets, suppose, $I : A \to Y$ and $J : B \to Z$ be functions satisfying, $I(A) \subset B$. If, I and J are differentiable at $x_0 \in A$ and $I(x_0)$ respectively, then, $J \circ I : A \to Z$ is differentiable at x_0, and,

$$(J \circ I)'(x_0) = J'(I(x_0)) \circ I'(x_0). \tag{1.6}$$

2. (Mean Value Theorem) A priori given a differentiable function, $I : A \to Y$ and, $x_0, x_1 \in A$, let us define,

$$[x_0, x_1] = \{\lambda x_0 + (1-\lambda)x_1\ :\ 0 \le \lambda \le 1\}$$

to be a *line segment* in A.. Then,

$$||I(x_1) - I(x_0)|| \le \sup_{x\in[x_0,x_1]} ||I'(x)||||x_1 - x_0||. \tag{1.7}$$

3. (Taylor's Formula) A priori given a function, $I : A \to Y$ being differentiable at some $x \in A$, let us define,

$$R(h) := I(x+h) - I(x) + I'(x)(h)$$

Then, by (1.5), the Remainder Term, $R(h)$ satisfies,

$$\lim_{h\to 0}\frac{||R(h)||}{||h||}=0, \qquad \text{i.e., } R(h)=o(||h||). \tag{1.8}$$

Theorem 1.4.3. *Suppose,* $\Omega \subset \mathbb{R}^n$ *be bounded and open, and,* $p > 1$*. Moreover, let* $g : \mathbb{R} \to \mathbb{R}$ *be a* C^1*-function satisfying,*

1. $|g(t)| \leq a + b|t|^p$,
2. $|g'(t)| \leq a + b|t|^{p-1}$

for some constants a, b. Also,

$$I(u) := \int_\Omega g(u(x)). \tag{1.9}$$

Then, $I \in C^1(L^p(\Omega);\mathbb{R})$*, and, for every* $u \in L^p(\Omega)$*,*

$$I'(u)\phi = \int_\Omega g'(u)\phi, \qquad \forall\ \phi \in L^p(\Omega).$$

Corollary 1.4.4. *Suppose, for any continuous function,* $f : \mathbb{R} \to \mathbb{R}$ *satisfying,*

$$|f(t)| \leq a + b|t|^p$$

for every, $1 < p \leq \frac{(n+2)}{(n-2)}$*. Moreover, let,* $F(t) = \int_0^t f(s)ds$ *be the primitive of* f*. If,* $I : H_0^1(\Omega) \to \mathbb{R}$ *has the following definition,*

$$I(u) := \int_\Omega F(u(x))dx.$$

Then, $I \in C^1(H^1_0(\Omega);\mathbb{R})$, *and,*

$$I'(u)\phi = \int_\Omega f(u)\phi \qquad \forall\ \phi \in H_0^1(\Omega). \tag{1.10}$$

2 Critical Points

In this section, we assume everywhere unless mentioned otherwise, that, X is a *Banach Space*, and $A \subset X$.

Definition 2.0.1. (Minima of a Function) Given a function, $I : A \to \mathbb{R}$, we define a point $x_0 \in A$ to be a local minimum (resp. strictly local minimum) of I if, $\exists\ U(x_0)$ of x_0 in X satisfying,

$$I(x_0) \leq I(x) \qquad \forall\ x \in U(x_0) \cap A \tag{2.1}$$

resp.,

$$I(x_0) < I(x) \qquad \forall\ x \in U(x_0) \cap A \setminus \{x_0\} \tag{2.2}$$

On the other hand, x_0 is defined to be a global minimum of I if,

$$I(x_0) \leq I(x) \qquad \forall\ x \in A \tag{2.3}$$

Therefore, we write, $I(x_0) = \min_{x \in A} I(x)$.

$$I(x_0) \geq I(x) \qquad \forall\ x \in U(x_0) \cap A \tag{2.4}$$

resp.,

$$I(x_0) > I(x) \qquad \forall\ x \in U(x_0) \cap A \setminus \{x_0\} \tag{2.5}$$

On the other hand, x_0 is defined to be a global maximum of I if,

$$I(x_0) \geq I(x) \qquad \forall\ x \in A \tag{2.6}$$

Therefore, we write, $I(x_0) = \max_{x \in A} I(x)$.

Remark 2.0.1. Assume $A \subset X$ to be open, and $I : A \to \mathbb{R}$ be *diffeentiable* on A. Then,

$$I'(x_0) = 0$$

provided, $x_0 \in A$ is a local minima (or, a local maxima) of I.

Definition 2.0.3. (Critical Point) Suppose, $A \subset X$ be open, and $I : A \to R$ be differentiable on A. A point $x_0 \in A$ is defined to be a critical point of I if, $I'(x_0) = 0$. Subsequently, $I(x_0) \in \mathbb{R}$ is called the critical value.

It is evident from the definition that, *local minima* and *local maxima* of a differentiable function I on some subset $A \subset X$ are indeed *critical points* of I. However, the same cannot be infered about the converse of this result.

For example, we take, $X = \mathbb{R}^2$, $I(x, y) = x^2 - y^2$. Then, $I'(x, y) = (2x, -2y)$, and, $I'(0, 0) = 0$. Thus, $(0, 0)$ is a critical point of I, although, is neither a local minima nor a local maxima of I.
It follows from the fact that,

$$I(0,0) = 0\ ,\ I(x,0) = x^2 > 0 \text{ for, } x \neq 0,$$

$$I(0,y) = -y^2 < 0 \text{ for, } y \neq 0$$

***Remark* 2.0.2.** Critical points satisfying the condition as above as termed as *Saddle Points*.

Definition 2.0.4. (Saddle Point) Suppose, $A \subset X$ be open, and $I : A \to \mathbb{R}$ be differentiable on A. We define a critical point $x_0 \in A$ of I to be a saddle point of I if, for every neighbourhood $U(x_0)$ of x_0, $\exists\ x_1, x_2 \in U(x_0)$ satisfying,

$$I(x_1) < I(x_0) < I(x_2) \tag{2.7}$$

Definition 2.0.5. (Convex Function) A function $I : X \to \mathbb{R}$ is defined to be convex, if $\forall\ x, y \in X$,

$$I(tx + (1-t)y) \leq tI(x) + (1-t)I(y)\ , \qquad \forall\ t \in (0,1)$$

***Example* 2.0.3.** Consider, $X = \mathbb{R}$. Then, the function, $f : \mathbb{R} \to \mathbb{R}$ defined as, $f(x) = e^x$ is convex everywhere on $\mathbb{R}$.

***Remark* 2.0.4.** In case when, I is *convex* on X, the *critical points* of I are in fact global minima.

Proposition 2.0.5. *For a convex and differentiable function $I : X \to \mathbb{R}$,*

$$I(x_0) \leq I(x) \qquad \forall\ x \in X,$$

$$i.e.,\ I(x_0) = \min_{x \in X} I(x).$$

Proposition 2.0.6. *Suppose*, $I : X \to \mathbb{R}$ *be a differentiable function satisfying the following condition*,

$$I(y) \leq I(x) + I'(y)(y - x) \qquad \forall\ x, y \in X$$

Then, I is convex.

Proof. To prove the result, we'll use the definition (2.0.5) of convexity.
Given that $I(y) \leq I(x) + I'(y)(y - x)$ for every $x, y \in X$.
For $\lambda \in [0, 1]$, consider $z = \lambda x + (1 - \lambda)y$.
By the given condition:

$$\begin{aligned} I(z) &\leq I(x) + I'(z)(z - x) \\ &= I(x) + I'(z)(\lambda x + (1 - \lambda)y - x) \\ &\leq I(x) + \lambda I'(z)(x - y) \end{aligned}$$

Similarly,

$$\begin{aligned} I(z) &\leq I(y) + I'(z)(z - y) \\ &= I(y) + I'(z)(\lambda x + (1 - \lambda)y - y) \\ &\leq I(y) + (1 - \lambda)I'(z)(x - y) \end{aligned}$$

Combining these inequalities:

$$\begin{aligned} I(z) &\leq I(x) + \lambda I'(z)(x - y) \\ &\leq I(x) + \lambda I'(z)(x - y) + (1 - \lambda)I'(z)(x - y) \\ &= \lambda I(x) + (1 - \lambda)I(y) + \lambda(x - y)I'(z) \end{aligned}$$

Since $\lambda \in [0, 1]$, $I(z) \leq \lambda I(x) + (1 - \lambda)I(y)$. Hence, $I(x)$ is convex.

Before we investigate the existence of *critical points* of a function, one needs to justify that, the definitions (2.0.1) and (2.0.2) of minima and maxima respectively are well-defined.

Definition 2.0.6. A priori, given X to be a *hausdorff* topological space, a function, $I : X \to \mathbb{R}$ is defined to be lower semi-continuous if $\forall$ $c \in \mathbb{R}$, the set $\{x \in X \mid I(x) \leq c\}$ is *closed*.

Proposition 2.0.7. (i) *Every continuous function is lower semi-continuous.*

(ii) If $A \subset X$ *is open, then,* χ_A *is lower semi-continuous*.

Proof. (i) We intend to show that for any continuous function $f : X \to \mathbb{R}$, where X is a topological space, the set $\{x \in X : f(x) > \alpha\} = A$ (say) is open for every $\alpha \in \mathbb{R}$. This implies that for any point x_0 in X, there exists a neighborhood U of x_0 such that $f(x) > \alpha$ for all x in U.

Let $\alpha \in \mathbb{R}$ be arbitrarily chosen. For any point $x_0 \in A$, since $f(x_0) > \alpha$, by the continuity of f, $\exists$ a neighborhood U_{x0} of x_0 such that $f(x) >$ $\alpha\ \forall\ x \in U_{x0}$.

Thus, for any $x_0 \in A$, there exists a neighborhood U_{x0} of x_0 contained in A, implying A is open.

Therefore, every continuous function $f : X \to \mathbb{R}$ is lower semi-continuous.

(ii) To prove that the indicator function χ_A of an open set $A \subset X$ is lower semi-continuous, it suffices to show that for any $x_0 \in X$, and for any $\alpha < 1$, $\exists$ a neighborhood U of x_0 such that $\chi_A(x) > \alpha$ for all $x \in U$.

Formally, let χ_A be the indicator function of A, defined as:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Let $x_0 \in X$ be arbitrary, and let $\alpha < 1$ be given. Since A is open, $\exists$ a neighborhood U of x_0 contained in A. For any $x \in U$, $x \in$ A, so $\chi_A(x) = 1$. Since $\alpha < 1$, $\chi_A(x) > \alpha$ for all $x \in U$.

Therefore, χ_A of an open set $A \subset X$ is indeed lower semi-continuous.

***Remark* 2.0.8.** A function which is *lower semi-continuous* may not be continuous. For example, we can consider $X = \mathbb{R}$, and, $A = (a, b) \subset \mathbb{R}$ for any $a, b \in \mathbb{R}$ with $a < b$. Then, χ_A is indeed lower semi-continuous on R (using above proposition), although it is not continuous at the points a and b in $\mathbb{R}$.

Theorem 2.0.9. *A priori given X to be a compact and Hausdorff topological space. Furthermore, $I : X \to \mathbb{R}$ be lower semi-continuous. Then, I is bounded below, and, $\exists\, x_0 \in X$ such that,*

$$I(x_0) = \min_{x \in A} I(x).$$

Proof. I being lower semi-continuous implies that, the set, $A_n = \{x \in X \mid I(x) > -n\}$ is open $\forall\, n \in \mathbb{N}$. Moreover, $X = \bigcup_{n=1}^{\infty} A_n$, and, is compact. Hence, we must have, $X = \bigcup_{n=1}^{n_0} A_n$ for some $n_0 \in \mathbb{N}$.

As a consequence, $I(x) > -n_0\ \forall\, x \in X$. Thus, it establishes that, X is bounded below.
Let us choose, $\inf_X I = l\ (> -\infty)$. We claim that, such a "l" do exists. If not, then, we consider the collection $\{B_n\}_{n=1}^{\infty}$ where, $B_n :=$ $x \in X \mid I(x) > l + 1/n$ for every $n \in \mathbb{N}$ such that,

$$X = \bigcup_{n=1}^{\infty} B_n$$

A priori from the *compactness* condition of X, $\exists\, n_1 \in \mathbb{N}$ satisfying, $X = \bigcup_{n=1}^{n_1} B_n$. In other words, the collection, $\{B_n\}_{n=1}^{n_1}$ indeed is a finite subcover of X for some $n_1 \in \mathbb{N}$.

Subsequently, $I(x) > l + \frac{1}{n_1}$, $\forall\, x \in X$, a contradiction to the fact that, $l = \inf_X I$.

For the ease of computation, we introduce the notion of sequentially lower semi-continuous.

Definition 2.0.7. Given a *Hausdorff* topological space X, a function, $I : X \to \mathbb{R}$ is defined to be sequentially lower semi-continuous if, for every sequence $\{x_n\}$ tending to x in X,

$$I(x) \leq \varliminf_{n \to \infty} I(x_n) \tag{2.8}$$

Proposition 2.0.10. *For a Hausdorff topological space X and a function $I : X \to \mathbb{R}$, the following holds true:*

(1) *I is lower semi-continuous $\implies$ I is sequentially lower semi-continuous*.
(2) *Converse holds true only if X is a metric space*.

Proof. (1) Given $I : X \to \mathbb{R}$ to be lower semi-continuous. Suppose, (x_n) be a sequence in X converging to x, and suppose $I(x_n) \to I(x)$. We wish to show that $I(x) \le \liminf_{n\to\infty} I(x_n)$. Since I is lower semi-continuous, for any $\epsilon > 0$, $\exists$ a neighborhood U of x such that $I(y) > I(x) - \epsilon \ \forall \ y \in U$.

Since $x_n \to x$, $\exists \ N \in \mathbb{N}$ such that $\forall \ n \ge N$, $x_n \in U$.

Thus, for every $n \ge N$, $I(x_n) > I(x) - \epsilon$.

Taking $n \to \infty$,

$$\liminf_{n\to\infty} I(x_n) \ge I(x) - \epsilon$$

Since ϵ was arbitrary, we conclude:

$$I(x) \le \liminf_{n\to\infty} I(x_n)$$

Therefore, I is sequentially lower-semi continuous, and the proof is thus complete.

(2) Assume $I : X \to \mathbb{R}$ be a sequentially lower semi-continuous function, and let $\alpha \in \mathbb{R}$ be arbitrary. It suffices to establish that, the set $A := \{x \in X : I(x) > \alpha\}$ is open in X.
Let x_0 be any point in the set, say, $\{x \in X : I(x) > \alpha\}$, i.e., $I(x_0) > \alpha$.

Since I is sequentially lower semi-continuous, for any sequence (x_n) in X converging to x_0, we have $I(x_0) \le \liminf_{n\to\infty} I(x_n)$.

Since x_0 is in the set $\{x \in X : I(x) > \alpha\}$, we have $\liminf_{n\to\infty} I(x_n) > \alpha$.

As a result, $\exists \ N \in \mathbb{N}$ such that $\forall \ n \ge N$, $I(x_n) > \alpha$ [$\because$ the limit inferior of a sequence is the greatest lower bound of the set of subsequential limits].

Therefore, for any sequence (x_n) in X converging to x_0, $\exists$ a neighborhood U of x_0 such that $I(x) > \alpha$ for all $x \in U$, which implies that $\{x \in X : I(x) > \alpha\}$ is open in X.

Since α was arbitrary, this holds for all $\alpha \in \mathbb{R}$. Therefore, we can conclude that, I is lower semi-continuous.

***Remark* 2.0.11.** A *sequentially lower semi-continuous function* on a non metrizable space may not be lower semi-continuous. Consider the example where, $X = \mathbb{R}$ is equipped with the cofinite topology.

Define the function $f : X \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Consider any sequence (x_n) in X converging to x. Since every neighborhood of x in the cofinite topology contains all but finitely many points of X, x_n equals x for all sufficiently large n. Therefore, $\liminf_{n\to\infty} f(x_n) = f(x)$, which makes $f(x)$ sequentially lower semi-continuous.

Let's examine the point $x = 0$. The set $\{x \in X : f(x) > 0\}$ is the set of irrational numbers in X, which is not open in the cofinite topology because it contains infinitely many points. Hence, $f(x)$ is not lower semi-continuous.

Proposition 2.0.12. *Given a Hausdorff topological space X and, $I : X \to \mathbb{R}$, suppose that, for every $c \in \mathbb{R}$,*

$$\textit{The set } \{x \in X \ : \ I(x) \leq c\} \textit{ is } \textbf{\textit{compact}}, \tag{2.9}$$

Then, I is bounded below, and, $\exists\, x_0 \in X$ *satisfying*, $I(x_0) = \inf_{x \in X} I(x)$.

Proof. To prove this statement, we'll use the fact that compactness of the level sets of I implies certain properties about I.

Suppose I is not bounded below. Then, for each $n \in \mathbb{N}$, $\exists\, x_n \in X$ such that $I(x_n) < -n$. However, this implies that the set $\{x \in X : I(x) \leq -n\}$ is nonempty and contains the sequence $\{x_n\}$, but it cannot be compact as the sequence has no convergent subsequence due to the unboundedness of I. This contradicts the assumption that all such sets are compact. Hence, I must be bounded below.

Since I is bounded below, let $c = \inf_{x \in X} I(x)$. Then, for each $n \in \mathbb{N}$, $\exists\, x_n \in X$ such that $I(x_n) < c + 1\ n$. Consider the sequence $\{x_n\}$. Since $\{x \in X : I(x) \leq c + 1/n\}$ is compact, there exists a subsequence $\{x_{nk}\}$ converging to some x_0 in X. By continuity of I, $I(x_{nk}) \to I(x_0)$. But since $I(x_{nk}) < c + 1/n_k$, we have $I(x_0) \leq c$. Conversely, since c is the greatest lower bound of I, we have $I(x_0) \geq c$. Therefore, $I(x_0) = c$, and x_0 is the desired point.

***Remark* 2.0.13.** Consider, $X = \mathbb{R}$. Define a function, $I : \mathbb{R} \to \mathbb{R}$ as, $I(x) = e^{-x}$. We can observe that, I is indeed smooth and bounded below, but I does not achieve its infimum. Applying Theorem (2.0.9) and Proposition (2.0.10) in order to obtain infimum, a compactness condition either for the space or, for the function must be required.

In case for infinite dimensional Banach Space X, the compactness condition is not achieved under the norm topology. However a certain amount of compactness is achieved to ensure the attainment of the infimum can be obtained in weaker topology on X.

In the next section, we shall learn more about a weaker topology called *weak topology* on X as compared to the norm topology defined on it.

3. Weak Topology on Banach Spaces

3.1 Weak Convergence

Suppose we consider $(X, \|.\|)$ to be a Banach Space with X^* as its *dual*. Moreover, τ be the metric topology on X induced by the norm $\|.\|$, having the following definition, $d(x, y) = \|x - y\|$, $\forall\, x, y \in X$.

We intend to define the weakest topology τ_w on X as follows : "*Every functional $f \in X^*$ is continuous on X with respect to the topology τ_w on X*". The topology τ_w thus formed is defined as the Weak Topology on X. As for convergence of any sequence under *weak topology*, we provide the following definition.

Definition 3.1.1. (Weak Convergence) A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to converge to $x \in X$ weakly, and is denoted by, $x_n \rightharpoonup x$, if,

$$f(x_n) \longrightarrow f(x) \qquad \forall\ f \in X^*$$

***Remark* 3.1.1.** If a sequence $\{x_n\}_{n=1}^{\infty}$ converges to x with respect to the norm $\|.\|$ (i.e., $\|x_n - x\| \to 0$ as $n \to \infty$), we assert that, x_n converges to x strongly in X, in other words, $x_n \to x$.

***Proposition* 3.1.2.** *We can accumulate some important properties of weak convergence of sequences as follows*:

1. $x_n \rightharpoonup x$ *in* $X \Longrightarrow x$ *is unique*.

2. $x_n \to x$ *in* $X \Longrightarrow x_n \rightharpoonup x$. *The converse is not true in general*

3. $x_n \rightharpoonup x \Longrightarrow \{\|x_n\|\}_{n=1}^{\infty}$ *is bounded, and*, $\|x\| \leq \liminf_{n \to \infty} \|x_n\|$.

This implies, $\|.\| : X \to \mathbb{R}$ *is weakly sequentially lower semi-continuous*.

4. *Suppose, X be reflexive. Furthermore*, $\|x_n\| \leq M\ \forall\ n \in \mathbb{N}$ *for some*, $M > 0$. *Then*, $\exists\, x_0 \in X$ *and a subsequence*, $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ *in X such that*, $x_{n_k} \rightharpoonup x_0$.

5. *Let, Y be another Banach Space and $T \in B(X, Y)$. Then,*

- $x_n \rightharpoonup x \Rightarrow Tx_n \rightharpoonup T_x$.
- *If T is compact and,* $x_n \rightharpoonup x \Rightarrow Tx_n \longrightarrow T_x$ *in* Y.

6. *If X be a Hilbert Space, Then,* $x_n \rightharpoonup x$ *and,* $||x_n|| \longrightarrow ||x|| \Rightarrow x_n \longrightarrow x$.

Proof. 1. Suppose, $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that, $x_n \rightharpoonup x$ and, $x_n \rightharpoonup y$. Then, for every $f \in X^*$, $f(x_n) \to f(x)$ and, $f(x_n) \to f(y)$. SInce, $\{f(x_n)\}$ is a sequence of numbers, hence its limit is unique, i.e., $f(x) = f(y)$, i.e., for every $f \in X^*$, we have, $f(x - y) = 0$.

Therefore, using Corollary (4.3.4) (cf. [24, Pg 223]), we conclude, $x - y = 0$, and thus, the weak limit is indeed unique.

2. By definition, $x_n \longrightarrow x$ means, $||x_n - x|| \longrightarrow 0 \Rightarrow$ For every $f \in X^*$,

$$|f(x_n) - f(x)| = |f(x_n - x)| \leq ||f||.||x_n - x|| \longrightarrow 0.$$

Implying that, $x_n \rightharpoonup x$.

Converse is not true in general.

Consider the sequence $\{x_n\}$, where, $x_n = e_n$, $\forall\ n \in \mathbb{N}$ in ℓ^2, where e_n is the sequence whose n-th term is 1 and all other terms are 0. Let f be any bounded linear functional on ℓ^2. Then $f(e_n) = 1$ for all n. Since en is the unit vector along the n-th coordinate, $|f(e_n)| = \|f\|\|e_n\| = \|f\|$ (by the Cauchy-Schwarz inequality). Therefore, $f(e_n) = 1$ for all n implies that f does not converge to 0 as $n \to \infty$. Hence, $x_n = e_n$ converges weakly to 0. For strong convergence, we need to show that $\|x_n - 0\| = \|e_n\| \to 0$ as $n \to \infty$. However,
$\|e_n\| = 1$ for all n, so $\|x_n - 0\| = 1$ for all n, which does not tend to 0.
Therefore, the sequence xn in ℓ^2 converges weakly to 0 but does not converge strongly.

3. Given, $x_n \rightharpoonup x$. Thus, $f(x_n) \to f(x)\ \forall\ f \in X^* \Rightarrow \{f(x_n)\}$ is a convergent sequence of numbers, hence is bounded.

Let, $|f(x_n)| \leq c_f\ \forall\ n \in \mathbb{N}$, where, c_f is a constant depending on f, but not on n. Using the canonical mapping, $C: X \longrightarrow X^{**}$ (cf. (5) of Sec. 4.6 [24, Pg. 240]), where X^{**} denotes the double dual of X, we can in fact define, $g_n \in X^{**}$ by,

$$g_n(f) = f(x_n), \qquad f \in X^*$$

Then,

$$|g_n(f)| = |f(x_n)| \leq c_f \qquad n \in \mathbb{N}.$$

Implying that, the sequence, $\{|g_n(f)|\}$ is bounded for every $f \in X^*$. Since, X^* is complete, by (2.10.4) (cf. [24, Pg. 120]), we can apply the Uniform Boundedness Theorem (cf. [24, Pg. 249]) to conclude that, $\{||g_n||\}$ is bounded.

Now, $||g_n|| = ||x_n||$ by (4.6.1) (cf. [24, Pg. 240]) helps us conclude that, $\{||xn||\}$ is bounded.

As for the second part, if, $x = 0$, then, $||x|| = 0$, and the statement is obviously true. Now, we assume, $||x|| \neq 0$. By Theorem (4.3.3) (cf. [24, Pg. 223]), $\exists$ some $f \in X^*$ such that,

$$||f|| = 1\ , \quad f(x) = ||x||$$

Since, $\{x_n\}$ converges weakly to x, and, f is indeed continuous, we have,

$$\lim_{n\to\infty} f(x_n) = f(x) = ||x||.$$

But, $f(x_n) \leq |f(x_n)| \leq ||f||.||x_n|| = ||x_n||$. Hence,

$$\liminf_{n\to\infty} ||x_n|| \geq \lim_{n\to\infty} f(x_n) = ||x||.$$

4. To prove this statement, we can utilize the *Eberlein–Šmulian Theorem*, which states that in a reflexive Banach space, every bounded sequence has a weakly convergent subsequence. Since X is reflexive, every bounded sequence in X has a weakly convergent subsequence.

Let be a bounded sequence in X, i.e., $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. By the Eberlein–Smulian Theorem, ∃ a subsequence such that $x_{n_k} \rightharpoonup x_0$ in X. Therefore, $x_{n_k} \rightharpoonup x_0$ weakly, where $x_0 \in X$ and is a subsequence of .
The proof is thus complete.

5. Given, $x_n \rightharpoonup x$ in X, thus, $f(x_n) \longrightarrow f(x)$ for every $f \in X^*$.We intend to show that,

$$\varphi(T(x_n)) \longrightarrow \varphi(T(x)) \qquad \forall\ \varphi \in Y^*$$

In other words,

$$(\varphi \circ T)(x_n) \longrightarrow (\varphi \circ T)(x) \qquad \forall\ \varphi \in Y^*$$

Although, $\varphi \circ T \in X^*$, therefore, our hypothesis guarantees our desired conclusion.
• Suppose T is a compact operator from X to Y, and $x_n \rightharpoonup x$ in X implies $Tx_n \rightharpoonup Tx$ in Y. By the definition of compact operators, every bounded sequence $\{x_n\}$ in X has a weakly convergent subsequence $\{x_{n_k}\}$ in X. Suppose, $x_{n_k} \rightharpoonup x$. Since T is compact, $\{Tx_{n_k}\}$ has a convergent subsequence $\{Tx_{n_{k_j}}\}$ in Y. Let $Tx_{n_{k_j}} \to y$ in Y.

Now, we have $x_{n_{k_j}} \rightharpoonup x$ and $Tx_{n_{k_j}} \to y$. Since weak convergence implies boundedness, we have $\{x_{n_{k_j}}\}$ is bounded in X. By the first part of the statement, $Tx_{n_{k_j}} \rightharpoonup Tx$ in Y. But since $Tx_{n_{k_j}}$ converges to y in Y, by uniqueness of limits in Banach spaces, $Tx = y$. Hence, $Tx_n \to Tx$ in Y.

Therefore, if T is compact and $x_n \rightharpoonup x$ implies $Tx_n \to Tx$ in Y, and the statement holds true.

6. To prove this statement, let X be a Hilbert space, and suppose $x_n \rightharpoonup x$ weakly in X. Also, assume that $\|x_n\| \to \|x\|$.
Since $x_n \rightharpoonup x$ weakly, for any $y \in X$, we have $\langle x_n, y\rangle \to \langle x, y\rangle$.
Now, consider the sequence $y_n = x_n - x$. We have:

$$||y_n||^2 = \langle y_n, y_n\rangle = \langle x_n - x, x_n - x\rangle = ||x_n||^2 - 2\langle x_n, x\rangle + ||x||^2$$

Given that $\|x_n\| \longrightarrow \|x\|$, and $\langle x_n, y\rangle \longrightarrow \langle x, y\rangle$ for any y, it follows that $\langle x_n, x\rangle \longrightarrow \langle x, x\rangle$.

Thus, $\|y_n\|^2 = \|x_n\|^2 - 2\langle x_n,x\rangle + \|x\|^2 \longrightarrow \|x\|^2 - 2\langle x,x\rangle + \|x\|^2 = 0$ as $n \to \infty$.
This implies that $\|y_n\| \to 0$. But $\|y_n\| = \|x_n - x\|$, so we have $x_n \to x$ in X.
Therefore, if X is a Hilbert space, $x_n \rightharpoonup x$ weakly and $\|x_n\| \longrightarrow \|x\|$, then $x_n \to x$ strongly in X.

3.2 Existence of Minima

Theorem 3.2.1. *Given a reflexive Banach Space* $(X, \|.\|)$, *suppose, A be a weakly sequentially closed subset of X. Define, I: A → R satisfying the following*:

(*I*) (*Compactness Condition*) *I is coercive on A, i.e.,*

$$I(u) \to \infty \qquad as\ ||u|| \to \infty,\ u \in A.$$

(*II*) *I is weakly sequentially lower semi-continuous on A, i.e., if* u_n, $u \in A$ *with* $u_n \rightharpoonup u$ *in X, then,*

$$I(u) \leq \liminf_{n\to\infty} I(u_n).$$

Then, I is indeed bounded below, and, $\exists\ u_0 \in A$ *such that*,

$$I(u_0) = \min_{u\in A} I(u).$$

Proof. Let us denote, $l := inf\ \{I(u) \mid u \in A\}$. We intend on proving that, $l > -\infty$ and, $I(u_0) = l$ for some $u_0 \in A$. Suppose, $\{u_n\} \subset A$ satisfying, $I(u_n) \to l$.

A priori using the fact that, I is coercive on A, $\exists\ M > 0$ such that, $\|u_n\| \leq M\ \forall\ n \in \mathbb{N}$.

Since, X is reflexive, $\exists$ a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ with $u_{n_k} \rightharpoonup u_0$ in X for some $u_0 \in X$.

Moreover, A being weakly sequentially closed, we thus conclude that, $u_0 \in A$. Also, I is given to be weakly sequentially lower semi-continuous on A, which yields,

$$l \leq I(u_0) \leq \liminf_{n\to\infty} I(u_{n_k}) = l = \inf_{u\in A} I(u).$$

And the proof is complete.

3.3 Applications of the Existence of Minima

3.3.1 Application in Linear PDEs

We can in fact utilize Theorem (3.2.1) in order to find solutions to linear partial differential equations in the following manner.

Theorem 3.3.1. *Suppose,* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. For every* $f \in L^2(\Omega)$ *(more generally,* $f \in H^{-1}(\Omega)$*),* $\exists$ *a weak solution,* $u_0 \in H_0^1(\Omega)$ *to the following problem*,

$$\begin{cases} -\Delta u = f & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases} \tag{3.1}$$

In other words, for every $\phi \in H^1_0(\Omega)$,

$$\int_\Omega \nabla u_0 \nabla \phi = \int_\Omega f\phi.$$

3.3.2 Constrained Minimization

A priori given a *Hilbert Space* X over $\mathbb{R}$ and $f,g \in C^1(H, \mathbb{R})$, let us define, $G := \{u \in H \mid g(u) = 0\}$. Moreover, let, $g'(u) \neq 0\ \forall\ u \in H$ (This implies that, G is a *manifold* of co-dimension 1).

An important observation is, the *gradient* of g, denoted as $g'(u)$ is in fact normal to G.

Correspondingly, the *tangent space* T_u at $u \in G$ is defined as,

$$T_u := \{v \in H \mid \langle g'(u), v\rangle = 0\}.$$

Definition 3.3.1. A point $u_0 \in G$ is defined to be a **critical point** of $(f|G)$, i.e., $(f|G)'(u_0) = 0$ if,

$$f'(u_0)(v) = 0 \qquad \forall\ v \in T_u.$$

T_{u0} being of co-dimension 1, we conclude that, $f'(u_0) = \mu g'(u_0)$, for some $\mu \in \mathbb{R}$.
μ in the above case is defined to be the **Lagrange Multiplier**.

Proposition 3.3.2. *If* $\exists\ u_0 \in G$ *and,* $f(u_0) = \min_{u\in G} \{f(u)\} \in G$, then, we have,

$$(f|_G)'(u_0) = 0.$$

3.3.3 Application in Non-Linear PDEs

Assume $\Omega \subset \mathbb{R}^n$ be a bounded domain. We choose, $\lambda \in \mathbb{R}$ and, $1 < p \leq \frac{(n+2)}{(n-2)}$. We intend to obtain a weak solution to the following non-linear Dirichlet boundary value problem,

$$\begin{cases} -\Delta u = |u|^{p-1}u + \lambda u & \text{in } \Omega \\ u \geq 0,\ u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

In other words, $\forall\ \phi \in H^1_0(\Omega)$,

$$\int_\Omega \nabla u_0 \nabla \phi = \int_\Omega = \int_\Omega |u|^{p-1} u_0 \phi + \lambda \int u_0 \phi. \tag{3.3}$$

Taking, $\phi = u_0$ in (3.3),

$$\int_\Omega |\nabla u_0|^2 = \int_\Omega = \int_\Omega |u|^{p+1} + \lambda \int u_0^2. \tag{3.4}$$

Applying the **Sobolev Embedding Theorem**, we can in fact obtain the following embedding, $H^1_0(\Omega) \to L^q(\Omega)$, where, $q \leq \frac{2n}{(n-2)}$. Moreover, $q = p + 1$ helps us conclude that, there indeed exist a *weak solution* to the equation (3.2) in $H^1_0(\Omega)$ for every $1 < p \leq \frac{(n+2)}{(n-2)}$.

Suppose, $\phi \in H^1_0(\Omega)$ with $\phi \geq 0$ be the *Eigenfunction* corresponding to the first *Dirichlet Eigenvalue*, $\lambda_1(\Omega)$ of $-\Delta$ for the DIrichlet boundary condition as described in (3.2), having the following expression,

$$\lambda_1(\Omega) := \inf_{\phi \in H_0^1(\Omega)\setminus\{0\}} \left\{ \frac{\int_\Omega |\nabla \phi|^2}{\int_\Omega \phi^2} \right\}$$

Furthermore, from (3.2),

$$\begin{cases} -\Delta \phi = \lambda_1(\Omega)\phi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

Proposition 3.3.3. *For* $1 < p \leq \frac{(n+2)}{(n-2)}$ and, $\lambda \geq \lambda_1(\Omega)$, *the problem* (3.2) *admits no solution*.

Sobolev Embedding Theorem ensures the existence of an embedding, $H^1_0(\Omega) \hookrightarrow L^{p+1}(\Omega)$, which is compact for every $p < \frac{2n}{(n-2)}$. We shall comment later on the case when, $p = \frac{(n+2)}{(n-2)}$, $n \geq 3$.

Theorem 3.3.4. *Assume,* $1 < p < \frac{(n+2)}{(n-2)}$. *For every* $\lambda \in (-\infty, \lambda_1(\Omega))$, *the Dirichlet boundary value problem* (3.2) *admits of a weak solution. In other words*, $\exists\ u_0 \in H^1_0(\Omega)$ satisfying,

$$\int_\Omega \nabla u_0 \nabla \phi = \int_\Omega |u_0|^{p-1} u_0 \phi + \lambda \int u_0 \phi. \tag{3.6}$$

for every $\phi \in H^1_0(\Omega)$.

Proof. Proof of the above result primarily hinges on two claims. First, we shall introduce some notations.

We define, $J : H^1_0(\Omega) \to \mathbb{R}$ as,

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} - \frac{\lambda}{2}\int_\Omega u^2.$$

Moreover, for $g \in C^1(\mathbb{R})$ given by, $g(t) := |t|^{p-1}t + \lambda t$, and, $G(t) := \int_0^t g(s)ds$, we can in factinfer that,

$$G(u) = \frac{1}{p+1}\int_\Omega |u|^{p+1} + \frac{\lambda}{2}\int_\Omega u^2$$

and, also, $G \in C^1(H^1_0(\Omega),\mathbb{R})$. Thus, $J \in C^1(H^1_0(\Omega),\mathbb{R})$ as well and alsio for every $u \in H^1_0(\Omega)$ and $\forall\ \phi \in H^1_0(\Omega)$,

$$J'(u)\phi = \int_\Omega \nabla u \nabla \phi - \int_\Omega |u|^{p-1}u\phi - \lambda\int_\Omega u\phi. \tag{3.7}$$

Which suggests that, if $u_0 \in H^1_0(\Omega)$ with $J'(u_0) = 0$, then u_0 should satisfy (3.6). Hence, it only suffices to check for *critical points* of J.

Choose $0 \neq u_1 \in H^1_0(\Omega)$. Therefore, for any $t \in \mathbb{R}$, $tu \in H^1_0(\Omega)$ as well, and,

$$J(tu_1) = \frac{t^2}{2}\int_\Omega |\nabla u_1|^2 - \frac{t^{p+1}}{p+1}\int_\Omega |u_1|^{p+1} - \frac{\lambda t^2}{2}\int_\Omega u_1^2.$$

A priori fromt he fact that, J is unbounded below on $H^1_0(\Omega)$, in other words, $J(tu_1) \to -\infty$ as $t \to \infty$, we conclude that, J does not admit any minimum on $H^1_0(\Omega)$.

To find critical points of J, we denote,

$$A := \left\{ u \in H^1_0(\Omega) \mid \int_\Omega |u|^{p+1} = 1 \right\}$$

and, $I : A \to \mathbb{R}$ as,

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega u^2.$$

As a result, using Theorem (3.2.1), we state our first claim.

Lemma 3.3.5. *For any* $\lambda \in (-\infty, \lambda_1(\Omega))$, *I is bounded on A. Further,* $\exists\ u_\lambda \in A$ *satisfying,*

$$I(u_\lambda) = \min_{u \in A} I(u).$$

On the other hand, considering,

$$g(u) := \int_\Omega |u|^{p+1} - 1$$

and, subsequently, we define, $A := \{u \in H^1{}_0(\Omega \mid g(u) = 0)\}$. Hence, applying the concepts of *constrained minimization* as mentioned in Definition (3.3.1), we can in fact, establish our second claim.

Lemma 3.3.6. *For every* $\lambda \in (-\infty, \lambda_1(\Omega))$, $\exists\ c_\lambda > 0$ *such that,*

$$J'(c_\lambda u_\lambda) = 0.$$

In this case, we can in fact deduce that, $c_\lambda = (\mu(p+1))^{1/(p-1)}$, for some $\mu \in \mathbb{R}_+$.

Therefore, $\tilde{u}_\lambda := c_\lambda u_\lambda$ is a weak solution of (3.2). Furthermore, $u_\lambda \geq 0$, $u_\lambda \neq 0$ implying, $\tilde{u}_\lambda \geq 0$, and the proof is complete.

Remark 3.3.7. For $p = \frac{(n+2)}{(n-2)}$, the embedding, $H^1{}_0(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is not compact. Under this scenario, the corresponding Dirichlet boundary value problem (3.2) is termed as the problem with lack of compactness, or, the critical exponent problem. As a result, the set A as defined above during our second claim need not be weakly sequentially closed, and thus our desired result is not achieved. In fact, the existence of weak solution in this case depends stricly on the choice of λ and the geometry of Ω

Theorem 3.3.8. (***Pohozaev's Identity***) *A priori given a locally lipschitz function,* $f : \mathbb{R} \to \mathbb{R}$ *with,* $\Omega \subset \mathbb{R}^n$ *being a bounded domain with smooth boundary. Moreover, let* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *satisfies,*

$$\begin{cases} -\Delta u = f & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases} \tag{3.8}$$

Define, $F(t) := \int_0^t f(s)ds$. *Suppose also that,* $\nu(x)$ *is the unit outward normal at* $x \in \partial\Omega$ *and,* $\frac{\partial u}{\partial \nu} = \nabla u.\nu$. *Under these assumptions, we have the following identity,*

$$2n \int_\Omega F(u) - (n-2) \int_\Omega f(u)u = \int_{\partial\Omega} x.\nu \left(\frac{\partial u}{\partial \nu}\right)^2 \tag{3.9}$$

As an important derivation from the above identity, we might infer the following.

Corollary 3.3.9. *Assuming,* $\Omega = B(R)$, *the following Dirichlet problem,*

$$\begin{cases} -\Delta u = u^{\frac{(n+2)}{(n-2)}} + \lambda u & in\ B(R) \\ u \geq 0,\ u \not\equiv 0 & in\ B(R) \\ u = 0 & on\ \partial B(R) \end{cases} \tag{3.10}$$

doesn't admit any solution in $C^2(\Omega) \cap C^1(\bar{\Omega})$ *for every* $\lambda \in (-\infty, 0]$.

Theorem 3.3.10. $\exists\ \lambda_1(\Omega) > 0$ *and,* $\phi_0 \in H_0^1(\Omega)$, $\phi_0 \geq 0$, $\phi_0 \not\equiv 0$ *satisfying,*

$$\lambda_1(\Omega) = \int_\Omega |\nabla \phi_0|^2$$

and,

$$\begin{cases} -\Delta\phi_0 = \lambda_1(\Omega)\phi_0 & in\ \Omega \\ \phi_0 \geq 0,\ \phi_0 \not\equiv 0 & in\ \Omega \\ \phi_0 = 0 & on\ \partial\Omega \end{cases} \tag{3.11}$$

Furthermore, $\forall\, \phi \in H_0^1(\Omega)$, *we must have*,

$$\lambda_1(\Omega)\int_\Omega \phi^2 \leq \int_\Omega |\nabla\phi|^2$$

Proof. We define,

$$S; = \left\{ \phi \in H_0^1(\Omega) \mid \int_\Omega \phi^2 = 1 \right\}$$

Hence,

$$\lambda_1(\Omega) = \inf_{\phi\in S}\left\{ \int_\Omega |\nabla\phi|^2 \right\}$$

Applying a result by *Poincare*, $\lambda_1(\Omega) > 0$. We choose a *minimizing sequence* $\phi_n \in S$, such that,

$$\int_\Omega |\nabla\phi_n|^2 \longrightarrow \lambda_1(\Omega).$$

Implying that, ϕ_n is bounded in $H^1{}_0(\Omega)$. Thus, $\exists$ a sunsequence, $\{\phi_{n_k}\}_{k=1}^\infty$ of $\{\phi_n\}_{n=1}^\infty$ with $\phi_{n_k} \rightharpoonup \phi_0$ in $H^1{}_0(\Omega)$. From Rellich's Theorem, $H^1{}_0(\Omega) \hookrightarrow L^2(\Omega)$ is indeed compact. Hence, $\phi_{n_k} \rightarrow \phi_0$ in $L^2(\Omega)$, and,

$$\lim_{k\to\infty}\int_\Omega \phi_{n_k}^2 = \int_\Omega \phi_0^2.$$

Now, $\phi_{n_k} \in S$ and, $\int_\Omega \phi_0^2 = 1 \Rightarrow \phi_0 \in S$.

Furthermore, $||.||_{H_0^1(\Omega)}$ being *weakly sequentially lower semi-continuous* implies,

$$\lambda_1(\Omega) \leq \int_\Omega |\nabla\phi_0|^2 \leq \liminf_{k\to\infty}\int_\Omega |\nabla\phi_{n_k}|^2 = \lambda_1(\Omega).$$

Therefore, $\lambda_1(\Omega) = \int_\Omega |\nabla\phi_0|^2$.

Moreover, assuming $\phi_0 \geq 0$, $f(u) = \int_\Omega |\nabla u|^2$ and, $g(u) = \int_\Omega u^2 - 1$, where, $u \in S$, we obtain,

$$g'(u)u = 2\int_\Omega u^2 = 2 \neq 0.$$

A priori from the concepts discussed in the section of *Constrained Minimization*, we get,

$$\int_{\Omega} \nabla \phi_0 \nabla \phi = 2\mu \int_{\Omega} \phi_0 \phi$$

μ being the *Lagrange Multiplier.* Setting $\phi = \phi_0 \in S$, we obtain, $\lambda_1(\Omega) = 2\mu$.

Applying (3.11),

$$\int_{\Omega} \nabla \phi_0 \nabla \phi = \lambda_1(\Omega) \int_{\Omega} \phi_0 \phi \ , \qquad \forall \ \phi \in H_0^1(\Omega).$$

We thus conclude that, $\lambda_1(\Omega)$ and ϕ_0 indeed solve (3.11).

For any, $u \in H_0^1(\Omega)$, $u \not\equiv 0$, defining, $v = \frac{u}{\left(\int_{\Omega} u^2\right)^{1/2}}$, it can be observed that, $v \in S$ as well, so by definition of $\lambda_1(\Omega)$,

$$\lambda_1(\Omega) \leq \int_{\Omega} |\nabla v|^2 = \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2}.$$

i.e.,

$$\lambda_1(\Omega) \int_{\Omega} u^2 \leq \int_{\Omega} |\nabla u|^2 \, .$$

And, hence the proof is complete.

4 Ekeland Variational Principle and its Applications

4.1 Variational Principle

Assume, $I \in C^1(X, \mathbb{R})$ to be bounded below. Ekeland Variational Principle yields a sequaence of *minimizers* of I over X with certain conditions, which enables us to conclude that, in various situations we can derive the infimum of I over X. Especially, this principle is quite beneficial for problems of minimax type.

Theorem 4.1.1. (***Ekeland Variational Principle-Strong Form***) *Given a complete metric space* (X, d), *and a lower semi-continuous function* $I \in C^1(X, \mathbb{R})$ *which is bounded below, for every* $\epsilon > 0$, $\lambda > 0$ *and,* $x_0 \in X$ *satisfying,*

$$I(x_0) \leq \inf_X I + \epsilon,$$

$\exists \ \bar{x} \in X$ *such that, the following conditions hold true*:

(1) $I(\bar{x}) + \left(\frac{\epsilon}{\lambda}\right) d(\bar{x}, x_0) \leq I(x_0)$.

(2) $d(x_0, \bar{x}) \leq \lambda$.

(3) $I(\bar{x}) < I(x) + \left(\frac{\epsilon}{\lambda}\right) d(x, \bar{x}), \quad \forall \ x \in X \ , \ x \neq \bar{x}$.

Proof. Suppose, $d_1(x, y) := \frac{\epsilon}{\lambda} d(x, y)$. Then, (X, d_1) is complete. Moreover, for $x \in X$, we define,

$$G(x) := \{y \in X \mid I(y) + d_1(x, y) \leq I(x)\} \, .$$

The above definition yields the following:
(1) $x \in G(x)$ and, $G(x)$ is in fact closed.
(2) $G(y) \subset G(x)$ if, $y \in G(x)$.
(3) For $y \in G(x)$, we have, $d_1(x, y) \leq I(x) - v(x)$, where, $v(x) := \inf_{z \in G(x)} I(z)$.

A priori from the fact that, $I(.) + d_1(x, .)$ being *lower semi-continuous*, and by definition, we can assert that, $G(x)$ is indeed closed and, $x \in G(x)$.

Let, $z \in G(y) \Rightarrow I(z) + d_1(z, y) \leq I(y) \Rightarrow I(y) + d_1(y, x) \leq I(x)$. Hence, (2) follows from the *triangle inequality*. Subsequently, we can deduce (3) using definitions of $G(x)$ and $v(x)$. Starting from x_0, our objective is to construct a sequence, $\{x_n\} \subset X$ such that,

$$x_{n+1} \in G(x_n) \ni I(x_{n+1}) \leq v(x_n) + \frac{1}{2^n} \qquad \text{for } n \geq 0.$$

Since, $v(x_0) = \inf_{x \in G(x_0)} I(x)$, thus, $\exists x_1$ with,

$$x_1 \in G(x_0) \ni I(x_1) \leq v(x_0) + \frac{1}{2}.$$

Now, considering $v(x_1)$, we obtain x_2 and so on. Using the fact that, $x_{n+1} \in G(x_n)$ for every $n \geq 0$, and (2),

$$G(x_n) \supset G(x_{n+1}) \qquad \forall\ n \geq 0.$$

We can derive that,

$$diam\ G(x_{n+1}) = \sup\{d_1(x, y) \mid x, y \in G(x_{n+1})\} \longrightarrow 0 \quad \text{ as } n \to \infty.$$

Now, $\{G(x_n)\}_{n\geq 0}$ being a *decreasing sequence* of closed sets with diameter tending to 0, by Cantor's Intersection Theorem, we get,

$$\bigcap_{n=0}^{\infty} G(x_n) = \{\bar{x}\} \qquad \text{for some } \bar{x} \in X.$$

We claim that, $\bar{x}$ is the required pint satisfying all the conditions as described in the statement of the Theorem.

$$\bar{x} \in G(x_0) \implies I(\bar{x}) + d_1(x_0, \bar{x}) \leq I(x_0). \text{ Furthermore,}$$

$$d_1(\bar{x}, x_0) \leq I(x_0) - v(x_0) \leq \epsilon.$$

Important to observe that, $G(\bar{x}) = \{\bar{x}\}$. Then, for every $x \in X$, $x \neq \bar{x}$, we must have, $x \notin G(\bar{x})$. Consequently,

$$I(x) + d_1(x, \bar{x}) > I(\bar{x}).$$

That completes the proof, a priori using definition, $d_1 = \frac{\epsilon}{\lambda} d$.

For the case, $\lambda = 1$, we can have a weaker version as follows.

Theorem 4.1.2. (***Ekeland Variational Principle-Weak Form***) *Assuming (X, d) to be a complete metric space, and $I : X \to \mathbb{R}$ be lower semi-continuous and bounded below. For every chosen $\epsilon > 0$, $\exists\ x_\epsilon \in X$ satisfying,*

(I) $I(x_\epsilon) \leq \inf_{x \in X} I(x) + \epsilon.$

(II) $I(x_\epsilon) < I(x) + \epsilon d(x, x_\epsilon) \qquad \forall\ x_\epsilon \in X\ ,\ x_\epsilon \neq x.$

As a corollary, one can deduce the following.

Corollary 4.1.3. *Suppose, X be a Banach Space, and, $I \in C^1(X, \mathbb{R})$ be bounded below. Then, $\exists$ a sequence, $\{x_n\}_{n\geq 0}$ in X satisfying,*

$$I(x_n) \to \inf_{x\in X} I(x) \ \textit{and,}\ I'(x_n) \to 0 \ \textit{in}\ X^*.$$

Proof. Applying the weak form of *Ekeland Variational Principle* (Theorem (4.1.2)), a priori given any $\epsilon > 0$, $\exists\ x_\epsilon \in X$ satisfying,

$$I(x_\epsilon) \leq \inf_X I + \epsilon$$

$$I(x_\epsilon) < I(x) + \epsilon||x - x_\epsilon||, \quad \forall\ x \in X,\ x \neq x_\epsilon. \tag{4.1}$$

Choose, $x = x_\epsilon = ty$, $t > 0$, $y \in X$, $y \neq 0$.
(4.1) yields,

$$I(x_\epsilon) - I(x_\epsilon + ty) < \epsilon t||y||.$$

Thus,

$$\lim_{t\to 0} \frac{I(x_\epsilon) - I(x_\epsilon + ty)}{t} \leq \epsilon||y|| \quad \Longrightarrow -I'(x_\epsilon)(y) \leq \epsilon||y|| \ , \quad \forall\ y \in X$$

$$\Longrightarrow\ I'(x_\epsilon)(y) \leq \epsilon||y|| \ , \quad \forall\ y \in X \quad (\text{ changing } y \text{ to } -y)$$

Hence,

$$||I'(x_\epsilon)|| = \sup_{\substack{y\in X\\ y\neq 0}} \left\{ \frac{|I'(x_\epsilon)y|}{||y||} \right\} \leq \epsilon.$$

Now, taking $\epsilon = \frac{1}{n}$, $x_\epsilon = x_n$, we thus have,

$$\inf_X I \leq I(x_n) \leq \inf_X I + \frac{1}{n} \implies ||I'(x_n)|| \leq \frac{1}{n}.$$

Hence, we conclude our desired result using definition of strong convergence

4.2 Palais-Smale Condition

We begin with a proper definition of the above.

Definition 4.2.1. Assume X to be a Banach Space, and, $I \in C^1(X, \mathbb{R})$. For any $c \in \mathbb{R}$, we say that, I satisfies the Palais-Smale Condition at c ($(PS)_c$ as abbreviation) if, every sequence $\{x_n\}_{n\geq 0} \subset X$ with, $I(x_n) \to c$, $I'(x_n) \to 0$ in X^*, has a convergent subsequence.

If in fact, I satisfies $(PS)_c$ at every $c \in \mathbb{R}$, then, we conclude that, I indeed satisfies the Palais-Smale Condition ((PS) as in short).

Theorem 4.2.1. *X be a Banach Space, and, $I \in C^1(X, \mathbb{R})$ is bounded below. Furthermore, let, I satisfy $(PS)_c$, where, $c = \inf_X$. Then, $\exists$ x0 $\in X$ satisfying,*

$$I(x_0) = \inf_{x\in X} I(x) \ \textit{and,}\ I'(x_0) = 0.$$

Proof. Corollary (4.1.3) implies that, $\exists$ a sequence $\{x_n\}_{n\geq 0} \subset X$ such that, $I(x_n) \to \inf_X I = c$ (say), and, $I'(x_n) \to 0$ as $n \to \infty$. Since, I satisfies the $(PS)_c$, $\exists$ a subsequence, $\{x_{n_k}\}_{k\geq 0}$ of $\{x_n\}_{n\geq 0}$ and $x_0 \in X$ satisfying, $x_n \to x_0$ in X.

Now, $I \in C^1(X, \mathbb{R})$ and $I' \in C(X, X^*)$ both being continuous, we shall obtain,

$$I(x_0) = \lim_{k\to\infty} I(x_{n_k}) = c,$$

and,

$$I'(x_0) = \lim_{k\to\infty} I'(x_{n_k}) = 0.$$

and the result holds true.

Corollary 4.2.2. *Suppose, $\Omega \subset \mathbb{R}^n$ be bounded, and, $1 \le q < p <$. For every $\lambda \in \mathbb{R}$, we define, $I : H^1{}_0(\Omega) \to \mathbb{R}$* as,

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} - \frac{\lambda}{2}\int_\Omega |u|^{q+1} \tag{4.2}$$

Then, I satisfies the (*PS*) *condition.*

Corollary 4.2.3. *Let, $p = \frac{(n+2)}{(n-2)}$, and, I be defined as in* (4.2). *Then, I satisfies $(PS)_c$ for every $c \in \left(-\infty, \frac{1}{n}S^{n/2}\right)$, where,*

$$S = \inf\left\{\int_\Omega |\nabla u|^2 \ : \ u \in H_0^1(\Omega) \ and, \ \int_\Omega |u|^{2n/(n-2)} = 1\right\}.$$

S as defined above is termed as the Best Sobolev Constant. Moreover, I does not satisfy $(PS)_c$, where, $c = \frac{1}{n}S^{n/2}$.

Proof. • To prove that I satisfies $(PS)_c$ for $c = \frac{1}{n}S^{n/2}$, where S is the Best Sobolev Constant, we proceed as follows:

Given $c = \frac{1}{n}S^{n/2}$, let $\{u_n\} \subset H^1{}_0(\Omega)$ be a Palais-Smale sequence at level c. By definition, this means that $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$.

We aim to show that every Palais-Smale sequence at level c has a convergent subsequence. To do so, we will use the Mountain Pass Theorem.

The *Mountain Pass Theorem* states that if I satisfies certain conditions, including the Palais-Smale condition and coercivity, then it possesses a critical point at every level below the Mountain Pass value.

Now, since $c = \frac{1}{n}S^{n/2}$, it implies that c is below the Mountain Pass value γ. Therefore, by the Mountain Pass Theorem, every Palais-Smale sequence at level c has a convergent subsequence converging to a minimum of I.

Hence, I satisfies $(PS)_c$ for $c = \frac{1}{n}S^{n/2}$.

• To prove that $(PS)_c$ fails for $c = \frac{1}{n}S^{n/2}$, where S is the Best Sobolev Constant, we construct a Palais-Smale sequence $\{u_n\} \subset H^1{}_0(\Omega)$ at level c that does not have a convergent subsequence

Given $c = \frac{1}{n}S^{n/2}$, we construct a Palais-Smale sequence $\{u_n\}$ as follows: Define,

$$u_n(x) = \sqrt{\frac{2}{\lambda_1(\Omega)}}\sin(\lambda_n x_1)\sin(\lambda_n x_2)\cdots\sin(\lambda_n x_n),$$

where λ_n is the n-th eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

Each function u_n is an eigenfunction of $-\Delta$ with Dirichlet boundary conditions, normalized such that $\|u_n\| = 1$. Therefore, $I(u_n)$ achieves the desired level $c = \frac{1}{n}S^{n/2}$.

By the properties of eigenfunctions, $I'(u_n) = u_n - \lambda_1(\Omega)u_n = (1 - \lambda_1(\Omega))u_n$. Since $\lambda_1(\Omega) > 0$, $I'(u_n) \to 0$ as $n \to \infty$.

However, the sequence $\|u_n\|$ does not converge, as it remains constant for all n. This lack of convergence implies that there is no convergent subsequence, violating the Palais-Smale condition.

Hence, we've found a Palais-Smale sequence $\{u_n\}$ at level $c = \frac{1}{n}S^{n/2}$ that does not have a convergent subsequence. Therefore, I fails the Palais-Smale condition at $c = \frac{1}{n}S^{n/2}$.

• Given the functional $I(u)$ defined as:

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda_1(\Omega)}{2}\int_\Omega u^2$$

where λ_n is the n-th eigenvalue of $-\Delta$ with Dirichlet boundary conditions

Each function u_n is an eigenfunction of $-\Delta$ with Dirichlet boundary conditions, normalized such that $\|u_n\| = 1$. Therefore, $I(u_n)$ achieves the desired level $c = \frac{1}{n}S^{n/2}$.

By the properties of eigenfunctions, $I'(u_n) = u_n - \lambda_1(\Omega)u_n = (1 - \lambda_1(\Omega))u_n$. Since $\lambda_1(\Omega) > 0$, $I'(u_n) \to 0$ as $n \to \infty$.

However, the sequence $\|u_n\|$ does not converge, as it remains constant for all n. This lack of convergence implies that there is no convergent subsequence, violating the Palais-Smale condition.

Hence, we've found a Palais-Smale sequence $\{u_n\}$ at level $c = \frac{1}{n}S^{n/2}$ that does not have a convergent subsequence. Therefore, I fails the Palais-Smale condition at $c = \frac{1}{n}S^{n/2}$.

Given the functional $I(u)$ defined as:

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda_1(\Omega)}{2}\int_\Omega u^2$$

where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $-\Delta$ on Ω, we aim to derive the Best Sobolev Constant S, defined as:

$$S = \inf\left\{\int_\Omega |\nabla u|^2 : u \in H_0^1(\Omega), \int_\Omega |u|^{2n/(n-2)} = 1\right\}$$

Consider the functional $J(u) = \int_\Omega |\nabla u|^2$ subject to the constraint $\int_\Omega |u|^{2n/(n-2)} = 1$. By the Euler-Lagrange equation, the critical points of $J(u)$ subject to the constraint satisfy:

$$-\Delta u + \lambda |u|^{2n/(n-2)-2}u = 0$$

where λ is a Lagrange multiplier. Let u be a nontrivial solution of the above equation. By scaling, we may assume that $\|u\|_{L^{2n/(n-2)}} = 1$. Then, by the variational characterization of $\lambda_1(\Omega)$, we have:

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}$$

Since $\|u\|_{L^{2n/(n-2)}} = 1$, we have $\int_\Omega |u|^{2n/(n-2)} = 1$, implying that $\int_\Omega |u|^2$ achieves its maximum. Thus, $\lambda_1(\Omega)$ is the smallest eigenvalue. Therefore, the Best Sobolev Constant S is equal to the first Dirichlet eigenvalue $\lambda_1(\Omega)$ of $-\Delta$ on Ω.

Hence, $S = \lambda_1(\Omega)$. This completes the proof.

Proposition 4.2.4. *Suppose,* $\lambda_1(\Omega)$ *be the first Dirichlet Eigenvalue of* $-\Delta$. *Define,* $I : H^1_0(\Omega) \longrightarrow \mathbb{R}$ as,

$$I(u) := \frac{1}{2} \int\limits_{\Omega} |\nabla u|^2 - \frac{\lambda_1(\Omega)}{2} \int\limits_{\Omega} u^2.$$

Then, I does not satisfy Palais Smale Condition at the point 0.

Proof. Given the functional $I(u)$ defined as:

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\lambda_1(\Omega)}{2} \int_{\Omega} u^2$$

where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $-\Delta$ on Ω, we aim to demonstrate that I does not satisfy the Palais-Smale condition at the point 0.

To do so, we construct a sequence $\{u_n\} \subset H^1_0(\Omega)$ such that $I(u_n) \to 0$, $\|u_n\| \to \infty$, and $I'(u_n) \to 0$ as $n \to \infty$.

Consider the function,

$$u_n(x) = \left(\sqrt{\frac{2}{\lambda_1(\Omega)}}\right)^{\frac{1}{n}} \sin(\lambda_n x_1) \sin(\lambda_n x_2) \cdots \sin(\lambda_n x_n),$$

where $x = (x_1, x_2, \ldots, x_n)$ represents the spatial coordinates in Ω, and λ_n is the n-th eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

Now, let's justify the properties:

• Convergence of $I(u_n) \to 0$: Substituting un into $I(u)$, we have:

$$I(u_n) = \frac{1}{2}\lambda_n - \frac{\lambda_1(\Omega)}{2} \to 0$$

as $n \to \infty$, since λ_n is the n-th eigenvalue and $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ on Ω.

• Divergence of $\|u_n\| \to \infty$: The norm of u_n is given by:

$$\|u_n\| = \left(\sqrt{\frac{2}{\lambda_1(\Omega)}}\right)^{\frac{1}{n}} \to \infty$$

as $n \to \infty$.

Convergence of $I'(u_n) \to 0$: The derivative of $I(u)$ at un is given by:

$$I'(u_n) = (1 - \lambda_1(\Omega))u_n \to 0$$

as $n \to \infty$, since $\lambda_1(\Omega) > 0$.

Therefore, the sequence $\{u_n\}$ satisfies the conditions for the Palais-Smale condition to fail at the point 0. Thus, the functional I does not satisfy $(PS)_0$.

Lemma 4.2.5. *Consider a Banach Space X and,* $I \in C^1(X, \mathbb{R})$. *Assume,* $\alpha = \lim_{\|u\| \to \infty} I(u) \in \mathbb{R}$. *Then,* $\exists$ *a sequence,* $\{u_n\}_{n \geq 0} \subset X$ *satisfying,* $\|u_n\| \to \infty$, $I(u_n) \to \alpha$ *and,* $I'(u_n) \to 0$.

Proof. A priori it is given to us that, $I \in C^1(X, \mathbb{R})$ on the Banach space X, with $\alpha = \lim_{\|u\| \to \infty} I(u) \in \mathbb{R}$.

Let $\epsilon > 0$ and $u_0 \in X$. Then, Ekeland's Variational Principle says that $\exists\ u \in X$ such that:

1. $I(u) \leq I(u_0)$,
2. $||u - u_0|| < \epsilon$,
3. $I(u) < I(v) + \frac{1}{2\epsilon}||u - v||^2$ for all $v \neq u$.

Now, we try to construct a sequence $\{u_n\}$ which'll satisfy the desired conditions. Choose $u_0 \in X$ arbitrarily and $\epsilon = 1$. By Ekeland's Variational Principle, there exists $u_1 \in X$ satisfying conditions (1), (2), and (3) with $\epsilon = 1$.

Now, recursively construct the sequence $\{u_n\}$ as follows: For $n \geq 2$, apply Ekeland's Principle with $\epsilon = 1/n$ and $u_0 = u_{n-1}$. This gives us un satisfying conditions (1), (2), and (3) with $\epsilon = 1/n$.

By our construction, the sequence $\{u_n\}$ satisfies the following properties:

• $I(u_n) \leq I(u_{n-1})$ for all n, implying that $\{I(u_n)\}$ is a decreasing sequence.

• $||u_n - u_{n-1}|| < 1/n$ for all n, so $\{u_n\}$ is a Cauchy sequence.

• By the completeness of X, $\{u_n\}$ converges to some $u \in X$.

Furthermore, since $\{I(u_n)\}$ is decreasing and bounded below by α, it converges to α. Also, since I is continuously differentiable, $I'(u_n)$ converges to $I'(u) = 0$ as $n \to \infty$.

Assume if possible that, $||u_n|| \nrightarrow \infty$. Then, there exists $M > 0$ such that $||u_n|| \leq M$ for all n. But then $\lim_{n\to\infty} I(u_n) = -\infty$, contradicting the assumption that α is finite. Thus, $||u_n||$ must tend to infinity.

Therefore, we have constructed a sequence $\{u_n\}$ satisfying all the desired properties using Ekeland's Variational Principle. Thus the proof is complete.

As a consequence of Lemma (4.2.5), we can conclude the following.

Proposition 4.2.6. *If* $I \in C^1(X, \mathbb{R})$ *is bounded below, and satisfies the* (PS) *condition, then,* I *is coercive*.

4.3 Applications

We can in fact observe an application of *Ekeland Variational Principle* to derive critical point(s) at the minimax level.

Theorem 4.3.1. (*Brezis Theorem*) *Given a Banach Space* X *and,* $I \in C^1(X, \mathbb{R})$, *assume* K *to be a compact metric space. Moreover, let,* $K_0 \subset K$ *be a closed set, and,* $p_0 : K_0 \to X$ *be a continuous function.*

Define,

$$\Gamma := \{p \in C(K, X) \ni p|_{K_0} = p_0\}$$

and,

$$c := \inf_{p\in\Gamma} \max_{\xi\in K} I(p(\xi)) = \inf_{p\in\Gamma} \max_{x\in p(K)} I(x). \tag{4.3}$$

ALso, suppose that,

$$\max_{\xi\in K} I(p(\xi)) > \max_{\xi\in K_0} I(p_0(\xi)) \qquad \forall\ p \in \Gamma. \tag{4.4}$$

Then, $\exists$ *a sequence,* $\{x_n\}_{n\geq 0} \subset X$ *satisfying,* $I(x_n) \to c$ *and,* $I'(x_n) \to 0$ *in* X^* *as* $n \to \infty$*. Furthermore, if I satisfies* $(PS)_c$*, then,* $\exists\ x_0 \in X$ *with,* $i(x_0) = c$ *and,* $I'(x_0) = 0$*.*

Important to mention that, the proof mainly hinges upon the following result.

Lemma 4.3.2. (***Pseudo-Gradient Lemma***) *Assume any metric space* Y *, and* X *to be a Banach Space with,* $F \in C(Y, X^*)$*. Given any* $\sigma >$ 0, $\exists$ *a function,* $h : Y \to X$*, locally Lipschitz such that,* $\forall\ y \in Y$,

- $||h(y)||_X \leq 1.$
- $\langle F(y), h(y)\rangle \geq ||F(y)||_{X^*} - \sigma.$

Brezis Theorem can also be considered as a generalization of the Mountain-Pass Theorem introduced by *Ambrosetti* and *Rabinowitz*.

Theorem 4.3.3. (***Mountain Pass Theorem***) *A priori under the assumptions that, X be a Banach Space and,* $I \in C^1(X, \mathbb{R})$ *satisfies* (PS). *Moreover, we consider the following condition,*

$$\exists\ R > 0\ \textit{and},\ e \in X\ \textit{ such that},\ ||e|| > R\ \textit{and},\ b = \inf_{x\in\partial B_R(0)} I(x) > \max\{I(0), I(e)\}. \qquad (4.5)$$

Then, $\exists\ x_0 \in X$ *satisfying,* $I'(x_0) = 0$ *and,* $I(x_0) = c \geq b$*, where, we can in fact characterize c as,*

$$c = \inf_{p\in\Gamma} \max_{t\in[0,1]} I(p(t)).$$

such that,

$$\Gamma = \{p \in C((0,1]\,, X)\ :\ p(0) = 0\ ,\ p(1) = e\}\,.$$

Proof. A priori from the definition, we have, $c < \infty$. Thus, for every $p \in \Gamma$, $p([0, 1])\cap\partial B_R(0) \neq \phi$. As a result,

$$\max_{t\in[0,1]} I(p(t)) = \max_{x\in p[0,1]} I(x) \geq \inf_{x\in\partial B_R(0)} I(x) = b \qquad (4.6)$$

Therefore, $c \geq b$. To establish the existence of critical point, we need to utilize the hypothesis of Brezis Theorem. Let, $K = [0, 1]$, $K_0 = \{0, 1\}$, $p_0(0) = 0$ and, $p_0(1) = e$.

Now then it remains for us to verify the condition (4.4). Although, from (5.1), we obtain,

$$\max_{t\in[0,1]} I(p(t)) \geq \inf_{x\in\partial B_R(0)} I(x) = b > \max\{I(0), I(e)\}.$$

Applying *Brezis Theorem*, we conclude that, $\exists\ x_0 \in X$ satisfying, $I(x_0) = c$ and, $I'(x_0) = 0$. This completes the proof.

Remark **4.3.4.** The statement of the *Mountain Pass Theorem* need not be true in case if I does not satisfies (PS).

For example, we can look into the function, $I \in C^1(\mathbb{R}^2, \mathbb{R})$ defined as,

$$I(x, y) := x^2 - (x - 1)^3 y^2.$$

SInce, $I(0) = 0$ and, 0 is in fact a *local minima*, we can choose $R > 0$ small enough such that, $b = \inf_{x\in\partial BR(0)} I(x) > 0$. Furthermore, given $e \in \mathbb{R}^2$ with, $|e| > \mathbb{R}$ and, $I(e) < 0$. It can be observed that, all the conditions of the *Mountain Pass Theorem* are satisfied, except that, I satisfies (PS).

Also, since $I(0) = 0$ and, 0 is indeed the only critical point of I, there $\nexists$ any $x_0 \in \mathbb{R}^2$ satifying, $I(x_0) \geq b > 0$ and, $I'(x_0) = 0$.

***Remark* 4.3.5.** Geometrically speaking, *Mountain Pass Theorem* implies that, if a pair of points in the graph of a function I are indeed separated by a mountain range, then $\exists$ a *mountain pass* containing a critical point of I.

Another version of the famous *Mountain Pass Theorem* can be found in [15].

Theorem 4.3.6. *Given a Banach Space X, and $I : X \to \mathbb{R}$ be a C^1 functional which indeed satisfies the (PS) condition. Suppose, S be a closed subset of X which disconnects X. Furthermore, given x_0, $x_1 \in X$ which belong to distinct connected components of $X \setminus S$, if I is bounded below in S, and in fact, the following condition is verified*:

$$\inf_S I \geq b \ and \ \max\{I(x_0), I(x_1)\} < b \tag{4.7}$$

Also, let,

$$\Gamma = \{f \in C((0,1], X) \ : \ f(0) = x_0 \ , \ f(1) = x_1\}.$$

Then, we shall have,

$$c = \inf_{f\in\Gamma} \max_{t\in[0,1]} I(f(t)) > -\infty \tag{4.8}$$

will be a critical value. In other words, $\exists$ $x_0 \in X$ satisfying,

$$I(x_0) = c \ , \ I'(x_0) = 0.$$

***Remark* 4.3.7.** The *connectedness* referred above is in fact *arc-wise connectedness*. Thus, $X \setminus S$ is indeed a union of open arcwise connected components (ref. [16, Pg-116]). Hence, x_0 and x_1 being in distinct components implies that, any arc in X connecting x_0 and x_1 intercept S.

For example, one can in fact consider X to be a *hyperplane* in X or, the boundary of an open set [in particular, the boundary of a ball].

5 Applications to the Critical Point Theory

Theorem 5.0.1. *Suppose*, $1 < q \leq p < \frac{(n+2)}{(n-2)}$ *, and*, $\lambda \in \mathbb{R}$*. Also, we consider $\Omega \subset \mathbb{R}^n$ to be a bounded domain. Then $\exists$ $u_0 \in H^1_0(\Omega)$ which is in fact a weak solution of the problem*,

$$\begin{cases} -\Delta u = |u|^{p-1}u + \lambda|u|^{q-1}u & in\ \Omega \\ u = 0 & on\ \partial\Omega \\ u \not\equiv 0 & in\ \Omega \end{cases} \tag{5.1}$$

in the sense that,

$$\int_\Omega \nabla u_0 \nabla \phi = \int_\Omega |u_0|^{p-1}u_0\phi + \lambda \int_\Omega |u_0|^{q-1}u_0\phi \qquad \forall\ \phi \in \mathcal{D}(\Omega). \tag{5.2}$$

Proof. We define, $I \in C^1(H^1_0(\Omega), \mathbb{R})$ as,

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} - \frac{\lambda}{q+1}\int_\Omega |u|^{q+1}.$$

Clearly, we can verify that, I satisfies (*PS*). Next we check the conditions for the *Mountain Pass Theorem*. We obtain,

$$I(0) = 0.$$

SInce, $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, $H_0^1(\Omega) \hookrightarrow L^{q+1}(\Omega)$, hence,

$$\int_\Omega |u|^{p+1} \le c||u||^{p+1} \quad , \quad \int_\Omega |u|^{q+1} \le c||u||^{q+1}$$

for some $c > 0$. Therefore,

$$I(u) \ge \frac{1}{2}||u||^2 - \frac{c}{p+1}||u||^{p+1} - \frac{c}{q+1}\lambda||u||^{q+1} = \left\{\frac{1}{2}||u|| - c_1||u||^p - c_2\lambda||u||^q\right\}||u||$$

For $||u|| = R$, we have,

$$I(u) \ge \left\{\frac{1}{2}R - c_1R^p - c_2R^q\right\}R$$

Since, $p, q > 1$, thus choosing R small enough such that,

$$I(u) > a, \text{ on } ||u|| = R \text{ for some } a > 0.$$

Let us take any $u_1 \in H_0^1(\Omega)$. Thus, $tu_1 \in H_0^1(\Omega)$ for any $t \in \mathbb{R}$. Now, from the fact that,

$$I(tu_1) \longrightarrow -\infty \qquad \text{as } t \to \infty.$$

Choosing t large enough such that, $||t_0u_1|| = |t_0|.||u_1|| > R$ and, $I(t_0u_1) < 0$, and, $e = t_0u$, and applying the Mountain Pass Theorem, we can assert that, $\exists\ u_0 \in H^1_0(\Omega) \ni I'(u_0) = 0$ and, $I(u_0) \ge a$.

It can also be derived that, $u_0 \neq 0$, as, $a > 0$, and the proof is thus complete.

On the other hand, *Brezis* and *Nirenberg* [28] has developed signifcant results in observing the model problem:

$$\begin{cases} -\Delta u = u^p + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \tag{5.3}$$

When $p = \frac{(n+2)}{(n-2)}$, $n \ge 3$ and, λ be any real constant. We can consider the following cases.

Theorem 5.0.2. *In case when, $n \ge 4$, the problem* (5.3) *indeed has a solution for every $\lambda \in (0, \lambda_1)$, where, λ_1 denotes the first eigenvalue of* $-\Delta$.

Moreover, the problem does not admit any solution if $\lambda \notin (0, \lambda_1)$ and, λ is starshaped.

Theorem 5.0.3. *For $n = 3$, the problem* (5.3) *turns out to be much more delicate. In this scenario, a complte solution exists only if Ω is a ball. Subsequently, we shall have that,* (5.3) *yields a solution iff $\lambda \in (1/4\lambda_1, \lambda_1, \lambda_1$ being the first eigenvalue of* $-\Delta$.

***Remark* 5.0.4.** In case when $p > \frac{(n+2)}{(n-2)}$, *Brezis* and *Nirenberg* [28] discusses the concept of commenting on the results related to existence of solutions to (5.3) using the notion of general Bifurcation Theory. For example, as mentioned by *Rabinowitz* [29], the problem (5.3) possesses a *component* $\mathfrak{C}$ of solutions (λ, u), which meets $(\lambda_1, 0)$ and which is *unbounded* in $\mathbb{R} \times L^\infty(\Omega)$.

Furthermore, if $p = \frac{(n+2)}{(n-2)}$ and $n \ge 4$, applying the result in Theorem (5.0.2), we can conclude that, the projection of C on the λ-axis does in fact contain the interval $(0, \lambda_1)$ (with appropriate modifications being done when $n = 3$, $p = 5$).

Furthermore, if $p = \frac{(n+2)}{(n-2)}$ and $n \geq 4$, applying the result in Theorem (5.0.2), we can conclude that, the projection of C on the λ-axis does in fact contain the interval $(0, \lambda_1)$ (with appropriate modifications being done when $n = 3$, $p = 5$).

As in another scenario when, $p > \frac{(n+2)}{(n-2)}$ and Ω is star-shaped, then the problem (5.3) has no solution for $\lambda \leq \lambda^*$, λ^* being some positive constant depending on Ω and p. This was explicitly derived by Rabinowitz [30] for the case when, $n = 3$, $p = 7$. One can in fact use similar argument in the general case by applying Pohozaev's Identity.

In greater generality as compared to the Dirichlet boundary value problem (5.3), *Brezis* and *Nirenberg* [28] also have dealt with the following problem in detail:

$$\begin{cases} -\Delta u = u^p + \lambda u^q & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \\ \quad u > 0 & \text{in } \Omega \end{cases} \tag{5.4}$$

Where, $p = \frac{(n+2)}{(n-2)}$ and, $1 < q < p$, $\lambda > 0$ being a constant. Considering different cases for n, we can conclude about the existence of solution (if any) for (5.4) in the manner as described below.

Theorem 5.0.5. *For* $n \geq 4$, *the problem* (5.4) *indeed admits of a solution for every* $\lambda > 0$.

Theorem 5.0.6. *In case when* $n = 3$ *and consequently,* $p = 5$, *we can assert the following about existence of solution to the DIrichlet problem* (5.4):

(*i*) *If* $3 < q < 5$, ∃ *solution to the problem for every* $\lambda > 0$.
(*ii*) *For* $1 < q \leq 3$, *solution does exists only for sufficiently large values of* λ.

Brezis Theorem allows us to conclude that, for any function $I \in C^1(X, \mathbb{R})$ which is bounded below, and satisfy (*PS*), ∃ $u_0 \in X$ satisfying, $I'(u_0) = 0$.

As for another application of Theorem (4.3.1), we next justify the existence of a *critical point* for I in case it is bounded below on a finite dimensional subspace of X.

Theorem 5.0.7. (***Saddle Point Theorem (Rabinowitz***) [1]) *Assume* X *to be a Banach Space and,* $I \in C^1(X, \mathbb{R})$ *satisfies* (*PS*). *Also suppose,* $V \neq 0$ *be a finite dimensional subspace of* X *and,* $X = V \oplus E$. *Furthermore, we consider that,* $\exists\, R > 0$, $\alpha, \beta \in \mathbb{R}$ *such that,*

$$\max_{x \in \partial B_R^V(0)} I(x) \leq \alpha < \beta \leq \inf_{x \in E} I(x). \tag{5.5}$$

Where,

$$B_R^V(0) = \{x \in V \ : \ ||x|| \leq R\} \qquad \textit{and,} \quad \partial B_R^V = \{x \in V \ : \ ||x|| = R\}.$$

Then, $\exists\, x_0 \in X$ *satisfying,* $I'(x_0) = 0$. *Moreover, the critical value,* $c = I(x_0) \geq \beta$, *which in fact, can be characterized as,*

$$c = \inf_{S \in \Gamma} \max_{u \in S} I(u). \tag{5.6}$$

Where,

$$\Gamma := \left\{ S = \phi(\bar{B}_R^V(0)) \ : \ \phi \in C\left(\bar{B}_R^V(0), X\right) \ \textit{and, } \phi|_{\partial B_R^V} = id \right\}$$

Proof of the above Rabinowitz Theorem requires an application of the *Topological Degree* in $\mathbb{R}^n$.

Definition 5.0.1. (Topological Degree) Suppose, $\Omega \subset \mathbb{R}^n$ be a bounded and open set. Given $\varphi \in C(\bar{\Omega})$ and, $p \in \mathbb{R}^n \setminus \varphi(\partial\Omega)$, the topological degree (or, *Brouwer Degree*), $d(\varphi, \Omega, p)$ is defined to be an integer satisfying the following properties:

(*I*) $$d(id, \Omega, p) = \begin{cases} 1 & \text{if } p \in \Omega, \\ 0 & \text{if } p \notin \bar{\Omega}. \end{cases}$$

(*II*) $d(\varphi, \Omega, p) \neq 0 = \Rightarrow \exists\, q \in \Omega$ such that, $\varphi(q) = p$.

(*III*) $d(\varphi, \Omega, p) = 0$ if, $p \notin \varphi(\bar{\Omega})$.

(*IV*) (Addition-Excision Property) If $\Omega_1, \Omega_2 \subset \Omega$ are open with $\Omega_1 \cap \Omega_2 = \phi$ and, $p \notin \varphi(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2))$, then,

$$d(\varphi, \Omega, p) = d(\varphi, \Omega_1, p) + d(\varphi, \Omega_2, p).$$

(*V*) If $\varphi : [0, 1] \times \bar{\Omega} \to \mathbb{R}$ and, $p : [0, 1] \to \mathbb{R}^n$ be continuous, and moreover, $p(t) \notin \varphi(t, \partial\Omega)\ \forall\, t \in [0, 1]$, then, $d(\varphi(t, .), \Omega, p(t))$ is independent of t.

(*VI*) $d(\varphi_1, \Omega, p) = d(\varphi_2, \Omega, p)$ whenever, $\varphi_1|\partial\Omega = \varphi_2|\partial\Omega$.

(*VII*) (Product Property) If Ω_j 's are bounded open sets in $\mathbb{R}^{n_j}$ for every $j = 1, 2$, and, φ_j and p_j are such that, $p_j \in \mathbb{R}^{n_j} \setminus \varphi_j(\partial\Omega_j)$, $j = 1, 2$. Then,

$$d(\varphi_1 \times \varphi_2, \Omega_1 \times \Omega_2, (p_1, p_2)) = d(\varphi_1, \Omega_1, p_1) d(\varphi_@, \Omega_2, p_2)$$

Remark **5.0.8.** Given $\Omega \subset \mathbb{R}^n$ to be bounded and open, $\varphi \in C^1(\bar{\Omega}, \mathbb{R}^n)$ and, $p \in \mathbb{R}^n \setminus \varphi(\partial\Omega)$, we can relate the theory of the *Brouwer Degree* with the existence and multiplicity of solutions of the equation,

$$\varphi(q) = p \tag{5.7}$$

Assuming $\varphi'(q)$ to be *non-singular* whenever (5.7) holds true. Then, the *Inverse Function Theorem* yields, (5.7) can have only a finite number of solutions in Ω. In this so called "nice" case, the corresponding Brouwer Degree of φ with respect to Ω and p, denoted by $d(\varphi, \Omega, p)$ has the following expression,

$$d(\varphi, \Omega, p) = \sum_{q \in \varphi^{-1}(p) \cap \Omega} sgn\{\det \varphi'(q)\},$$

where, det A denotes the determinant of a square matrix A.

Remark **5.0.9.** The notion of the *Brouwer Degree* can also be extended from "regular" to "singular" values of C^2-functions, and then to continuous functions on $\mathbb{R}^n$ [ref. [27]].

Remark **5.0.10.** The definition of *topological degree* as provided above as well as its properties can in fact be extended to an infinite dimensional space, in which case it is kown as Leray-Schauder Degree [ref. [27]].

Proof of Theorem (5.0.7):

Proof. For any $c \in \mathbb{R}$ and $I \in C^1(X, \mathbb{R})$, we define the following sets,

$$K := \{x \in X \ : \ I'(x) = 0\}$$

and,

$$K_c := \{x \in K \ : \ I(x) = c\}.$$

First, we intend to prove that, $K_c \neq \phi$ for c as mentioned in (5.6). Assume if possible that, $K_c = \phi$ in this case. We choose ϵ as,

$$0 < \epsilon < \frac{1}{4}(\beta - \alpha) \tag{5.8}$$

and, consider $S \in \Gamma$ such that,

$$\max_{x \in S} I(x) < c + \epsilon. \tag{5.9}$$

Such that, $(PS)_c$ condition is satisfied under these assumptions. Let, $\eta : [0, 1] \times X \to X$ be a I-decreasing homotopy (ref. Corollary (1.7) [25, Pg. 27]) satisfying the following conditions,

- $\eta(t, x) = x$ if, $|I(x) - c| \geq 2\epsilon$.
- $\eta(1, I^{c+\epsilon}) \subset I^{c-\epsilon}$, where, $I^c := \{x \in X \ : \ I(x) \leq c\}$.

We denote, $S_1 = \eta(1, S)$. If $S_1 \in \Gamma$, then by (5.9) we derive,

$$\max_{x \in S_1} I(x) \leq c - \epsilon$$

which contradicts the definition of c.

Let us establish that, $S_1 \in \Gamma$. Consider $\phi \in C(\bar{B}_R^V(0), X)$ such that, $\phi|_{\partial B_R^V(0)} = id$ and, = $\phi(\bar{B}_R^V(0))$. We thus have, $\phi_1 = \eta(1, \phi) \in C(\bar{B}_R^V(0), X)$ and, $S_1 = \phi_1(\bar{B}_R^V(0))$. Hence, $S_1 \in \Gamma$ $\phi_1(x) = x, \quad \forall\, x \in \partial B_R^V(0)$.

We claim that,

$$c \geq \beta. \tag{5.10}$$

If (5.10) does hold true, then, for $x \in \partial B_R^V(0)$, using (5.5) and (5.8),

$$I(x) \leq \alpha < \alpha + 2\epsilon < \beta - 2\epsilon \leq c - 2\epsilon.$$

This helps us assert that, $\phi_1(x) = x$.

Therefore, it only suffices to show (5.10). Wednote P_1 and P_2 to be the projections of X onto V and E respectively. Furthermore, we identify V with $\mathbb{R}^n$ for some n. For $\phi \in C(\bar{B}_R^V(0), X)$, $\phi = id$ on $\partial B_R^V(0)$. Hence, by properties (I) and (VI) in the definition (5.0.1) of the *Topological Degree*, we obtain,

$$d(P_1\phi, B_R^V(0), 0) = d(id, B_R^V(0), 0) = 1.$$

Applying property (II) in definition (5.0.1), $\exists\, x_0 \in B_R^V(0)$ satisfying, $P_1\phi(x_0) = 0$. Consequently, for each $S = \phi(\bar{B}_R^V(0)) \in \Gamma$, $\exists\, x_0 \in B_R^V(0)$ such that,

$$\phi(x_0) = P_2\phi(x_0) \in E.$$

On the other hand, from (5.5), we can conclude,

$$\max_{x \in \bar{B}_R^V(0)} I(\phi(x)) \geq I(\phi(x_0)) \geq \beta.$$

Using (5.6), it follows that, $c \geq \beta$, and the proof is thus complete.

Remark 5.0.11 . Heuristically speaking, in the above Theorem (5.0.7), c is the *minimax* of I over all surfaces modelled on $B_R^V(0)$, sharing the same boundary. Unlike the *Mountain-Pass Theorem*, in applications of the *Saddle Point Theorem*, in general, no critical points of I are known initially. Important to note that, the condition (5.5) are satisfied if I is *convex* on E, *concave* on V, and appropriately *coercive*.

Another version of the *Rabinowitz Saddle Point Theorem* (Theorem (5.0.7)) can be found in [26].

Theorem 5.0.12. *Given a real Banach Space X having the following direct sum decomposition, $X = V \oplus E$, where V and E are closed subspaces with $\dim V < \infty$. Suppose, $I \in C^1(X, \mathbb{R})$ satisfy (PS) condition, and,*

$$I(x) \longrightarrow -\infty \ as \ ||x|| \to \infty \ for \ x \in X \tag{5.11}$$

and,

$$\inf_{y \in E} I(x) = d > -\infty \tag{5.12}$$

Define, $D := \{x \in V \ : \ ||x|| \leq R\}$ with R is chosen so large that, $x \in V$ and $||x|| = R \Longrightarrow I(x) < d$.

If,

$$\Gamma := \{\phi \in C(D, X) \ : \ \phi|_{\partial D} = id\} \tag{5.13}$$

Then

$$c = \inf_{\phi \in \Gamma} \max_{x \in D} I(\phi(x)) > -\infty \tag{5.14}$$

and, $\exists \ u_0 \in X$ satisfying, $I(u_0) = c$ and, $I'(u_0) = 0$.

***Remark* 5.0.13.** For other versions of proof of Theorem (5.0.7) and, other important applications of the Mountain Pass Theorem, one can refer to [1-25].

***Remark* 5.0.14.** Theorem (5.0.7) essentially states that under certain conditions, a functional (a function of functions) will have at least one critical point that is a saddle point. This critical point is where the functional doesn't increase or decrease, representing a sort of equilibrium.

To put it in a more formal geometric context, consider a functional defined on an infinitedimensional space. The space can be thought of as a "landscape" of all possible functions. The functional assigns a "height" (or value) to each function in this landscape. Rabinowitz's theorem guarantees that there's at least one function in this landscape that has a saddle point: it's not the highest or lowest point, but it's a point of balance between different "directions" in the function space. This geometric interpretation is quite abstract because we're dealing with spaces that are not easy to visualize. However, the concept of a saddle point as a point of equilibrium remains a powerful image to understand the essence of the theorem.

Statements and Declarations

Conflicts of Interest Statement

I as the author of this article declare no conflicts of interest.

Data Availability Statement

I as the sole author of this article confirm that the data supporting the findings of this study are available within the article [and/or] its supplementary materials.

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