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A Very Fast Reasoning in Favor of the Poincare Group, or of Minkowskian Rather than Euclidean Line Elements

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1. We present a very fast derivation of Minkowski's line element which is the geometric foundation not only of the special Lorentz transformation but of the whole Poincar'e group aka inhomogeneous Lorentz group [1]. The Euclidean line element is excluded using the wave equation and Einstein's special principle of relativity (Einstein's first postulate) [2]. For doing so, it is not necessary,

1. to assume the speed of light to be independent of the motion of the observer (Einstein's second postulate [2],
2. to synchronize watches,
3. to interrelate the coordinate systems of the inertial frames involved; their relative velocity (and hence reciprocity [3]) even plays not any role in the derivation.
4. to make any assumption about the properties of the underlying coordinate transformation such as linearity, uniqueness etc

(to name a very few among the vast amount of prepositions in the literature).

Thus, consider two 3+1-dimensional inertial systems, S and S', with equal Cartesian coordinate systems. The relative positions of them plays not any role. Further, there be two scalar waves of the same kind the amplitudes $u(\vec{r}, t)$ and $u'(\vec{r}', t')$ of which are described by Euler's aka d'Alembert's wave equation (cf. eq. (A.5)),

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u\,, \tag{1a}$$

$$\frac{1}{c'^2}\frac{\partial^2 u'}{\partial t'2} = \nabla'^2 u'\,. \tag{1b}$$

2. The retarded Green's functions of the wave equations (1) read

$$G(\vec{r}-\vec{r}_0, t-t_0) = \frac{1}{4\pi|\vec{r}-\vec{r}_0|}\,\delta\left(t-t_0-\frac{|\vec{r}-\vec{r}_0|}{c}\right), \quad t-t_0 \ge 0\,, \tag{2a}$$

$$G'(\vec{r}'-\vec{r}'_0, t'-t'_0) = \frac{1}{4\pi|\vec{r}'-\vec{r}'_0|}\,\delta\left(t'-t'_0-\frac{|\vec{r}'-\vec{r}'_0|}{c'}\right), \quad t'-t'_0 \ge 0\,. \tag{2b}$$

Since the origins of the coordinate systems of *S* and *S'* play no role, we set $\vec{r}'_0 = \vec{r}_0 = \vec{0}$ and $|\vec{r}| =: r$, $|\vec{r}'| =: r'$. This simplifies the Green's functions (2) to

$$G(\vec{r},t) = \frac{1}{4\pi r}\,\delta\Big(t - \frac{r}{c}\Big)\,, \quad t \geq 0\,, \tag{3a}$$

$$G'(\vec{r}',t') = \frac{1}{4\pi r'}\,\delta\Big(t' - \frac{r'}{c'}\Big)\,, \quad t' \geq 0\,. \tag{3b}$$

3. According to the principle of relativity, the propagation speeds c and c' are equal. For otherwise, one could distinguish the systems *S* and *S'* by measuring the speed of that waves within them. Hence, the coordinates of the two outgoing spherical waves described by the Green's function (3) satisfy the relation

$$t' - \frac{r'}{c} = t - \frac{r}{c} = 0\,, \quad t', t \geq 0\,. \tag{4}$$

Analogously, for incoming waves, one obtains from the advanced Green's functions

$$t' + \frac{r'}{c} = t + \frac{r}{c} = 0\,, \quad t', t \leq 0\,. \tag{5}$$

Combining eqs. (4) and (5) yields

$$(ct')^2 - r'^2 = (ct)^2 - r^2 = 0\,, \quad -\infty < t', t < +\infty\,. \tag{6}$$

Notice that this has nothing to do with Einstein's second postulate that the measured speed of light is independent of the velocity of the observer [2].

4. The wave equations (1) presupposes the mediums in which the waves propagate to be homogeneous and isotropic [4]. As a consequence, the only possible invariant lengths squared s^2 are the Euclidean s^2_E and Minkowskian s^2_M ones,

$$s_E^2 = (ct)^2 + (\vec{r})^2 \quad \text{and} \quad s_M^2 = \pm\big((ct)^2 - (\vec{r})^2\big)\,. \tag{7}$$

Now, by virtue of relation (6), we have, for the waves under consideration,

$$(s'_E)^2 = (ct')^2 + (\vec{r}')^2 \neq s_E^2 = (ct)^2 + (\vec{r})^2 \tag{8a}$$

$$(s'_M)^2 = \pm\big((ct')^2 - (\vec{r}')^2\big) = s_M^2 = \pm\big((ct)^2 - (\vec{r})^2\big) = 0\,. \tag{8b}$$

The invariant length and hence the metrics of spacetime are not Euclidean but Minkowskian. The set of all coordinate transformations against which $s^2 = s^2_M$ is invariant is the Poincare group. Einstein's special theory of relativity [2] (STR) assumes all physical laws to be invariant against Poincar'e transformations with c being the speed of light in vacuum.

5. Admittedly Einstein's and this derivations are purely kinematic ones. It holds for Maxwell's equations in vacuum [2-6]. The inertial force of a body can be generalized to a Lorentz-invariant form using Euler's derivation of Newton's equation of motion, in agreement with Planck's derivation of the Newton-Lorentz force law [7-10]. However, for more general dynamical systems, there may be deviations [11,12]. Hence, the validity of STR has to be proven by experiment, what is actually done, again and again, cf., e.g. [13,14].

Finally, it is not claimed that this derivation is faster than Macdonald's one, surely not for the special Lorentz transformation. Despite that is up to the readers, he, however makes much more assumptions than we do [15].

1. Einstein's second postulate (see above; it is sufficient that the speed of light is one and the same within all inertial systems as required by Einstein's principle of relativity (his first postulate in [2]));
2. spatial homogeneity and isotropy of all inertial frames in vacuum (for a critical remark, see [4]);
3. the reading of an inertial clock may depend on its speed in an inertial frame (we don't need the reading of an inertial clock from another inertial system).

Moreover, for generalizing his result to 3+1d as well as for establishing Minkowski's line element, Macdonald would need further steps.

Acknowledgment

After the collaboration of one of us (PE) with Dieter Suisky on this topic [7], his interest in it has been sparked, again, by Roman Rupp's book [17], see [5][6][18]. Moreover, this work has been benefited from discussions with Alan Macdonald [15][18] and Mikhail Verkhovski [5][19][20]. Special thank is due to the latter one for sending his book [20]. The core idea of exploiting the Green's function has been developed during his guest professorships in Arkalyk and Pavlodar, Kazakhstan, which were determined by great collaboration, friendship, and support. Last but not least, we feel highly indebted to the many volunteers who are making master pieces of the past and advice for LATEX available in the internet.

Appendix: A Reasoning for the wave Equation

The assumption of the wave equation in vacuum is the main reason for making this derivation so fast. For the sake of making no more assumptions than necessary (Ockham's razor), however, it is desirable to derive the wave equation from principles which need no new assumptions. Here, following [21][22], such an approach will be sketched. [19-23].

For demonstrating the principle, it is sufficient to consider the scalar density $\rho(x, t)$ in 1+1d of a conserved quantity. Then, the continuity equation holds,

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} \,. \tag{A.1}$$

In the simplest case, the current density $j(x, t)$ is caused by a gradient of the density $\rho(x, t)$,

$$\tau \frac{\partial j}{\partial t} + j = -D \frac{\partial \rho}{\partial x} \tag{A.2}$$

(D being a diffusion coefficient). The first term on the l.h.s. of this material equation, the relaxation term, is borrowed from Maxwell's model of viscoelasticity [23] (τ being the relaxation time). It slows down the propagation of changes of the density to a finite value as we will see next.

Differentiating eq. (A.1) w.r.t. t and eq. (A.2) w.r.t. x, one can eliminate j to obtain a kind of the telegrapher's equation, [24],

$$\tau \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial \rho}{\partial t} = -D \frac{\partial \rho}{\partial x} \,. \tag{A.3}$$

Finally, we make the limit transition

$$\tau, D \to \infty, \quad D/\tau = c^2, \quad c \text{ finite}, \tag{A.4}$$

to obtain d'Alembert's wave equation

$$\frac{\partial^2 \rho}{\partial t^2} = c^2 \frac{\partial^2 \rho}{\partial x^2} \,. \tag{A.5}$$

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