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Research Article

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A Technical Lemma on Unitary ($\mathfrak{g}$, K)-Modules

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Abstract
In a pre-hilbertian structure of an unitary ($\mathfrak{g}$, K) -module underlie spinor subspaces that are spin invariant modules under right and left actions of G and that are images of endomorphisms restricted on t belonging to the Lie algebra $\mathfrak{g}$.

Keywords: Pre-Hilbertian Structures, Spinor Space, Spinors, Unitary ($\mathfrak{g}$, K) Modules

1. Introduction

One condition inherent of those unitary ($\mathfrak{g}$, K)-modules whose endomorphisms are in $\mathfrak{g}$ is that these must be restricted to the algebra $\mathfrak{t} \subset \mathfrak{g}$, considering the Cartan decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$. However, the nature of their endomorphisms in the corresponding compact maximal torus **T**, which is isomorphic to the standard torus T and whose Lie group is an compact Abelian Lie subgroup of G, has spin modules. These can be related by tensor product and are finally subspaces characterized by spinors. Likewise, for example $T(\gamma(\upsilon)\otimes I + \gamma(\upsilon)\otimes I) = \gamma(\upsilon)T$, where γ is the Dirac operator $(\gamma : V \rightarrow \mathrm{End}(S))$, $\upsilon \in V$, and T is a linear isomorphism. Here I^+ is the identity mapping on U^+, or on U^-, where U^+ and U^- are subspaces of $S(V)$, the set of element $\gamma(\upsilon)$. In the technical lemma, we want establish that $\mathfrak{so}(V)$ is the compact image of all the endomorphisms of the Lie algebra $S(V)$ restricted to the subalgebra $\mathfrak{t}$ [1]. This has advantages to establish linear isomorphisms and define a restriction of $\mathfrak{so}(V)$ sobre $\wedge V$X.

2. Mean Lemma

Definition 2.1. *Let V be a vector space of finite dimension on $\mathbb{R}$ with inner product (,). Then a spinor space to $(V, (\; , \;))$ is determined for the pair*

$$\gamma(\upsilon)^2 = -(\upsilon, \upsilon)I, \quad \forall \upsilon \in V. \tag{1}$$

Let G be a real reductive group. Then, $S(V)$ is the *spinor space* of V.

$$S(V) = \{\gamma(\upsilon) \in \mathrm{End}(S) \mid \gamma(\upsilon)^2 = -(\upsilon, \upsilon)I, \; \forall \; \upsilon \in V\},$$

Definition 2.2. A ($\mathfrak{g}$, K) -module is *unitary* if there is a pre-hilbertian Structure (,) on V such that $\forall X \in \mathfrak{g}$, $k \in K$ and u, $w \in V$,

Lemma 2.1. *In $S(V)$ exists a pre-hilbertian structure $\langle \, , \rangle$ such that*

$\forall \upsilon \in V$ *and* $u, w \in S(V)$ *with* $\gamma(\upsilon) \in \mathrm{End}(S)$,

$$\langle \gamma(\upsilon)\upsilon, w \rangle = -\langle u, \gamma(\upsilon)\upsilon \rangle, \tag{2}$$

For proof, see and [2-4].

A pre-hilbertiana structure of an unitary ($\mathfrak{g}$, K)-module is the Hermitian structure given by a product or form (,).

Lemma 2.2 (F. Bulnes)**.** *In a pre-hilbertian structure of an unitary* ($\mathfrak{g}$, K)*-module can be constructed a spinor subspace whose endomorphisms are endomorphisms of the Lie algebra* $\mathfrak{g}$ *restricted to the algebra* $\mathfrak{t}$.

Proof. Let $\mathfrak{g}$ be a semi simple Lie algebra on p with Cartan involution θ and corresponding Cartan decomp osition $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$. Consider a vector space V of finite dimension with inner product (,). Let $S(V)$be the spinor space to (V, (,)).

If V=$\mathfrak{p}$, the lemma follows trivially, since the spinor module of S(p) whose pre-hilbertian structure given for $\langle , \rangle$ on $S(\mathfrak{p})$ is such that it is a subspace with a t -invariant inner product and the endomorphisms of End($S(\mathfrak{p})$) are images of End($S(\mathfrak{p})$) restricted to $\mathfrak{t}$.

If V =$\mathfrak{g}$, then we extend (,) to a X-bilineal form on V. However, the said X-bilineal form conforms to a pre-hilbertian structure on $S(V)$

(Lemma 2. 1), and thus of the spin module (μ, S *(V))* $\forall \mu \in \mathfrak{so}(V)^*$

considering $\mu \in \mathfrak{so}(V) \rightarrow$ End ($S(V)$)), which is a (g, K)-module in the pre-hilbertian space

$$S(V) = H(K) \otimes 1_{V^{\pm}}. \qquad (3)$$

By the demonstration of the Lemma [2], there exists a t -invariant inner product $\langle , \rangle$ on $S(V)$. Thus $\gamma(\upsilon)|_{\mathfrak{t}} = \gamma(\upsilon)\, X$ t ,, and then $\mathfrak{g} = \mathfrak{so}(V)$, where $\mathfrak{so}(V)$ is the compact image of endomorphism of the Lie algebra $S(V)$ restricted to the subalgebra t.

In particular, if ($\mathfrak{g}$, K) -module is unitary, said pre-hilbertian structure induced by the product (,) is a Hermitian form and is a sesquilineal form in each complex of the corresponding cohomology on $\wedge V\mathrm{X}$, that is to say, in each one of their restrictions on $\wedge V\mathrm{X}$, respect to $\mathfrak{t}$, these restrictions are the corresponding images of $\mathfrak{so}(V)$. Then can be constructed a spinor subspace $W \subset S(V)$ such that End (S ((W) $= \mathfrak{so}(V)$ with $S(W)$ an unitary ($\mathfrak{g}$, K)-module.

3. Applications of Lemma

Example 3. 1. Little representations as cuspidal forms and infinite dimensional representations (possibly some G-modules induced by hyperbolic G-orbits G_h) can be expressed by a spinor decomposition. A concrete application of this, we can see the works to the twistor transform applied to finite dimensional representations of SU (p, q) and of SU (2, 2) to the problem no solved of the globalization of finite dimensional representations to the study of the Universe and the extension of said representations to the case of infinite dimensional representations [6].

The first generalized twistor transform to group representations with the idea of conform group was constructed by [7] of certain class representations of $SU(p, q)$ called ladder representations. These representations are those that can be determined for analytic continuation of the discrete series. The classification of the corresponding unitary modules of maximum weight are given in [8] and others. Its unitarization was demonstrated firstly in [9] using spinor structure underlying in the prehilbertian structure of the spin modules. Rawnsley et *al*,develop a general harmonic theory to indefinite metrics which include more of the ladder representations. Finally, all the set of ladder representations to $SU(p, q)$ was constructed using L^2 -cohomology where the Penrose transform plays an important role. In the result obtained in the choice of a complex structure on an underlying homogeneous space is linked to the parameter for the representation, therefore any possible action of Weyl group is lost on the parameters as in the discrete series [10].

We consider the vector complex space **T** of a complex manifold **M**, whose complex structure of **T** is given by the Hermitian form Φ with signature (p, q) such that $p+q = N+1$, with dim **T**= $N+1$. We consider the Lie group underlying in the complex manifold defined for $G = SU(p, q)$, which is a subgroup of $SL(N+1, \mathbb{C})$ and which preserves Φ. The projective space $\mathbb{P} = \mathbb{P}^N(\mathbb{C})$ divide to G in three open G-orbits: $\mathbb{P}^+$, $\mathbb{P}^-$ and $\mathbb{P}^0$, where

$$\mathbb{P}^+ = \{\text{lines} \subset \mathbf{T} \mid \Phi|_{\text{lines}} \geq 0 \Leftrightarrow \Phi \geq 0, \Phi = 0, \text{ if } \Phi|_{\text{lines}} = 0\}, \tag{4}$$

And

$$\mathbb{P}^- = \{\text{lines} \subset \mathbf{T} \mid \Phi|_{\text{lines}} \leq 0 \Leftrightarrow \Phi \leq 0, \Phi = 0, \text{ if } \Phi|_{\text{lines}} = 0\}. \tag{5}$$

Then $\mathbb{P}^0$, is a real hypersurface in $\mathbb{P}$.

$\mathbb{P}^+$, and $\mathbb{P}^-$, are the open G-orbits that determine the construction of the sesquilineal appearing more simple $\langle , \rangle$ in $SU(p, q)$. Said sesquilineal appearing induces a prehilbertian structure whose restriction to germs of the sheaf o $(-n-p)$ corresponds to the $-n-p$ power of the tautological bundle of lines on $\mathbb{P}$) of the complex holomorphic bundle

$$\mathbf{T} \to SU(p, q) \cong SL(N+1, \mathbb{C})/SO(N+1, \mathbb{C}), \tag{6}$$

which determines an inner product (,) on the cohomological space $H^{p-1}(|\mathbb{P}^+|, \mathfrak{o}(-n-p))$, and $\forall \phi, \psi \in H^{p-1}(|\mathbb{P}^+|, \mathfrak{o}(-n-p))$, the image of the Dirac operator is the module $\delta(\phi \cup T\psi)$ which is the spin space spin $(N, 1)$. Then underlies a spinor subspace $\delta(\phi \cup T\psi)$ in the unitary $(\mathfrak{g}, K)$ -module $H^{p-1}(|\mathbb{P}^+|, \mathfrak{o}(-n-p))$.

Then a concrete application of the pre-hilbertian structure where underlie spinor subspaces (whose linear endomorphisms are the representations of $SU(p, q)$ is the following result:

Theorem 3.1. *Let* $\langle , \rangle$ *be the inner product on* $H^{p-1}(|\mathbb{P}^+|, \mathfrak{o}(-n-p))$ *positive defined. Then the subspace* $H^{p-1}(|\mathbb{P}^+|, \mathfrak{o}(-n-p))$ *contains classes of cohomology that are dense spinor subspaces in a Hilbert space H. Then are K-finite vectors to the representation of SU* (p, q) *on H.*

This statement is a version of the Eastwood theorem in [7], which affirms the same that this in the language of the spinor subspaces underlying in all pre-hilbertian structure of the unitary module $H^{p-1}(|\mathbb{P}^+|, \mathfrak{o}(-n-p))$. The last line of the Theorem 3.1 says that the endomorphisms of the algebra $\mathfrak{g} = \mathfrak{su}(p, q)$ restricted to $\mathfrak{t} = \mathfrak{so}(N+1, \mathbb{C})$, are endomorphisms of the spinor subspaces of $H^{p-1}(|\mathbb{P}^+|, \mathfrak{o}(-n-p))$. Theorem 3.1 is proved by twistor transform in [6].

Example 3. 2. Another example of application is consider unitary representations such that

$$H^i(\mathfrak{g}, K; V \otimes F^*) = \mathrm{Hom}_K(\wedge^\bullet \mathfrak{p}, V \otimes F^*), \tag{7}$$

where V, is admissible and unitary ($\mathfrak{g}$, K)-module with infinitesimal character $\chi_{\Lambda+\rho}$, $\mathfrak{p}$, is a compact component of $\mathfrak{g}$ ($\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$) and F, is an unitary module as ($\mathfrak{g}_u$, G_u)-module with $u \in (\wedge^{\cdot}\mathfrak{p})^*$ and with a Hermitian form <, >, such that $\forall\, u$, $\upsilon \in F$, is satisfied $g<u, \upsilon> = <gu, g\upsilon>$ $\forall\, g \in G_u$ [11].

Likewise, on $\wedge^{\bullet}\mathfrak{p}$, we introduce a corresponding inner product to the restriction B (bilinear form) to $\mathfrak{p}$. The restriction of B to $\mathfrak{p}$, is an image complex $B_i(\wedge^{\cdot}\mathfrak{p}, V \otimes F^*) = dC_{i-1}(\wedge^{\cdot}\mathfrak{p}, V \otimes F^*)$, considering the complex sequence

$$\ldots \to C^{i-1}(\mathfrak{g}, K; V \otimes F^*) \to C^{i}(\mathfrak{g}, K; V \otimes F^*) \to C^{i+1}(\mathfrak{g}, K; V \otimes F^*) \to \ldots \quad (8)$$

Then is given the cohomology $H^i(g, K; V \otimes F^*)$ [2, 3, 5]. Likewise, has been used strongly the structure of the complex $C^i(g, K; V \otimes F^*)$ determined by the functorial diagram where appears the restriction of the corresponding endomorphisms of Lie algebra $\mathfrak{g}$, to the Lie algebra $\mathfrak{t}$,

$$\begin{array}{ccc} \wedge^i(\mathfrak{g}/\mathfrak{t}) & \to & V \otimes F^* \\ \text{Id} \searrow & & \nearrow \text{Hom}_K \\ & \wedge^i\mathfrak{p}, & \end{array} \quad (9)$$

Then the lemma 2. 2, is satisfied.

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