

Research Article

Journal of Electrical Electronics Engineering

A Novel Path-Following Method for Time-Varying Optimizations with Optimal Parametric Functions

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Submitted: 2024, May 08; **Accepted:** 2024, Jun 05; **Published:** 2024, Jul 11

Citation: Amidzadeh, M. (2024). A Novel Path-Following Method for Time-Varying Optimizations with Optimal Parametric Functions. *J Electrical Electron Eng, 3*(4), 01-10.

Abstract

We address a broad category of nonlinear constrained optimization problems. We then reformulate it as a time-varying optimization using continuous-time parametric functions and derive a dynamical system to track the respective optimal solution. We then re-parameterize the dynamical system according to a linear combination of parametric functions. By applying the calculus of variations, we optimize these parametric functions to minimize the optimality distance. Consequently, we develop an iterative dynamic algorithm, termed as OP-TVO, to achieve efficient convergence to the solution. We compare the performance of the proposed algorithm with the prediction correction method (PCM) in terms of optimality and computational complexity. The results demonstrate that OP-TVO effectively competes with PCM for the given class of optimization problems, suggesting a promising alternative to PCM. This work also introduces a novel paradigm for solving parametric dynamic systems.

Keywords: Time-Varying Optimization Problem, Functional Optimization Problem, Prediction-Correction Method, Optimality Distance, Dynamical System

1. Introduction

Time-Varying Optimization (TVO) problems involve parametric optimization where the objective function, constraints, or both are expressed using continuously varying functions. This approach is used to determine the optimal trajectory of solutions in continuous-time optimization scenarios. Additionally, for problems where the optimal solution is known for a specific configuration, TVO can be utilized to extra-polate the solution to other settings of interest.

TVO is explored in the domain of parametric programming, where the optimization problem is parameterized using continuous parameters [1-5]. In prediction-correction, methods are developed to address nonlinear constrained TVO problems, providing a mechanism to track a solution trajectory with certain convergence guarantees [3]. A path-following procedure is proposed in to trace the solution path of a parametric nonlinear problem [4]. A quadratic programming is then employed which leads to some convergence properties for their method. In a pathfollowing approach is used to track the solutions of parametric nonlinear constrained programs through a semi-smooth barrier function [5].

TVO can also be viewed as an extension of time-invariant optimization problems for the extrapolation purposes [6-10]. Presents an interior-point method for optimization problems with time-varying objective and constraint functions, formulating a continuous-time dynamical system to track the optimal solution with bounded asymptotic tracking error [8]. In path following, methods are devised in the dual space to track the solutions of time-varying linearly constrained problems [9].

When it goes to the approach for solving TVO problems, prediction-correction schemes emerge as promising tracking algorithms [11-14]. It involves a dynamic-tracking mechanism, called the predictor step to track the solution trajectory over time, accompanied with a Newton based iterative mechanism, called corrector step to adjust the prediction errors. In a discretetime prediction-correction, approach is proposed to minimize unconstrained time-varying functions, analyzing the asymptotic tracking error to ensure convergence [6]. In predictioncorrection methods are introduced to track the optimal solution trajectory in the primal space with a bounded asymptotic error [7]. In prediction-correction, methods are devised in the dual space to track the solutions of time-varying linearly constrained problems [9].

In this paper, we study a category of nonlinear constrained the literature, optimization problems where the optimal solution is known perspectives. for a specific setting and needs to be determined for a target configuration. We utilize the concept of TVO to reformulate the No problem using parametric programming. We then investigate a set of parametric functions to optimize the optimality distance. Unlike approaches based on prediction correction methods, our focus is on a functional optimization problem to expedite
the convergence rate. Specifically, this paper differs from the convergence rate. Specifically, this paper differs from prediction-correction approaches by designing parametric functions to minimize the optimality distance of the solution, rather than focusing on correcting prediction errors $[15,16]$. The n

• We address a class of nonlinear constrained optimization We us problems by formulating them as time varying optimization $\frac{1}{2}$ problems using parametric functions. We then use a re- 2. Problem parametrization trick to show that the corresponding dynamical system can be represented based on a linear combination of the parametric functions. $u_{\rm eff}$

• We globally optimize the parametric functions by a functional optimization procedure, and develop an iterative algorithm with The problem is specifically minimum optimality distance. We call the devised algorithm *Community distance.* We can the devised algorithm described as:
Optimal Parametric Time-Varying Optimization (OP-TVO).

• We compare OP-TVO with prediction-correction method from

his paper, we study a category of nonlinear constrained the literature, from the optimality and computational complexity perspectives. Ny and computational complexity

computational complexity perspectives.

main contributions of this paper are listed as follows:
and all-zeros vectors, respectively. **Notations:** In this paper, we use lower-case a for scalars, holdface lower-case a for vectors and holdface unpercase. n using parametric programming. We then investigate a boldface lower-case a for vectors and boldface uppercase **A** for matrices. Further, A^{\top} is the transpose of A , $\|\hat{a}\|$ is the the approaches based on prediction correction methods, Euclidean norm of $\mathbf{a}, \nabla_{\alpha} g(\cdot)$ and $\nabla_{\alpha}^2 g(\cdot)$ are the gradient vector functional optimization problem to expedite and Hessian matrix of multivariate function $g(\mathbf{a})$ with respect hal optimization problem to expedite and Hessian matrix of multivariate function g(a) with respect
pecifically, this paper differs from to (w.r.t.) vector a, respectively. We show the components of a convergence rate. Specifically, this paper different from to (w.r.t.) vector **a**, respectively. We show the components of a letterty distance of the solution, Further, $\{a_n\}^N$ collects the components of vector a from $n = 1$ to ecting prediction errors [15,16]. The $n = N$. We use **I**, 1 and 0 to denote the identity matrix, all-ones and all-zeros vectors, respectively. respectively. The contraction of $[1, 1, 1]$. We show the component of a n-dimensional components of this paper are listed as follows: se a for scalars,
Idface unnercase r
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mization We use $a(\theta)$ to represent the derivative of $a(\theta)$ w.r.t. θ . T and σ class of nonlinear constrained optimization problems. The problems and problems. The problem involves and problem involves and problems. The problems and problems and problems and problems and problems and pr α at α and α and α represented constraint α .

2. Problem Statement II. PROBLEM STATEMENT

 \sim We globally optimize the parametric functions by a functions by a function procedure, and develops by a function

that the corresponding dynamical This paper studies a class of nonlinear constrained optimization problems. This paper studies a class of nonlinear constrained optimization problems. The problem involves an objective function *f*(·) : ℝ^{*N*} tric functions.
 $\rightarrow \mathbb{R}$, vector-valued constraint functions $h_m(x) = [h_{m,1},...,h_{m,N}]^T(x)$ varametric functions by a functional $\cdot \mathbb{R}^N \to \mathbb{R}^N$ and optimization variables $x_m \in \mathbb{R}^N$ for $m \in \{1,...,M\}$. The problem is specifically described as: ased on a linear combination of the problems. The problem involves an objective function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$, vector-valued constraint functions $h_m(\cdot) = [h_{m,1},...,h_{m,N}]^T(\cdot)$ $\ddot{}$ R^N → R^N and optimization variables x^m ∈ R^N for m ∈ {1,...,M}. The problem is specifically p^{\prime} : minimal p^{\prime}

$$
P_0: \qquad \min_{\{\boldsymbol{x}_m\}_{1}^M} \sum_{m=1}^M a_m f(\boldsymbol{x}_m)
$$
\n
$$
\text{s.t.} \quad \sum_{m=1}^M \boldsymbol{h}_m(\boldsymbol{x}_m) = \boldsymbol{u}, \tag{1}
$$

Where $u \in \mathbb{R}^N$. The optimal solution of Problem P_0 is known arise in constrained problems with an objective established from for given non-zero parameters $a_m - p_{0,m}$, with $m \in \{1,..., M\}$. different cost functions with distinct weights $\{a_m\}_1$.
The **aim** is then to find the optimal solution for non-zero target parameters $a_m = p_{\tau,m}$. for given non-zero parameters $a_m = p_{0,m}$, with $m \in \{1,..., M\}$.

The aim is thus to obtain agent-specific variables $\{x_m\}_1^{u_1}$ that parameter optimize this overall cost function. This class of challenges also that Such problems arise in distributed optimizations or multi-agent systems, where individual agents are required to optimize agent-specific rewards contributing to an overall cost function. agent-specific rewards contributing to an overall cost function. Infictions. Consequently, we parameter
The aim is thus to obtain agent-specific variables $\{x_m\}_1^M$ that parametric functions $\{b_m(\theta)\}_1^M$ and parametriz

distinct weights {am}^M

 \overline{a}

M]

required to optimize a specific reduction of the aim is the aim is the aim is the aim is thus to an overall cost functions with distinct weights $\{a_m\}_{1}^M$. arise in constrained problems with an objective established from

relatively convergence rate being optimized. For this, we follow a t-specific rewards contributing to an overall cost function. functional optimization approach to design the parameterize $\{a_m\}_1^M$ with the If \mathcal{X}_m $\}_{1}^{\infty}$ that parametric functions $\{b_m(\theta)\}_{1}^{\infty}$ and parameter $\theta \in \mathbb{R}$ \cup $\{0\}$, so thallenges also that We then intend to express P_0 based on a TVO problem with gents are required to optimize functional optimization approach to design the parametric butting to an overall cost function. Functions. Consequently, we parameterize $\{u_{m}\}\$ with the gent-specific variables $\{x_{m}\}\$ that parametric functions $\{b_{m}(\theta)\}\$ ^M and parameter $\theta \in \mathbb{R}^{+} \cup \{0\}$, so parametric functions, and devise a path following method that also arise in construction with an objective established from different cost functions with an objective established from different cost functions with an objective established from different cost functions with an objecti $\{h_n\}_{n=1}^M$ that parametric functions $\{b_m(\theta)\}_{n=1}^M$ and parameter $\theta \in \mathbb{R}^+ \cup \{0\}$, so that

$$
\lim_{\theta \to 0} b_m(\theta) = p_{0,m}, \qquad \lim_{\theta \to \tau} b_m(\theta) = p_{\tau,m}, \qquad (2)
$$

 $W = (1, 10, M, d, d, d, 1, 1)$ based on a TVO problem with parametric functions, and device a pathfunctions that optimize the convergence rate. For $m \in \{1,...,M\}$. Note that such parametric functions as presented in (2) are not unique. However, we aim to find those parametric where the find the convergence rate. \mathbf{R} then consider the following TVO problem consider the following TVO problem \mathbf{R}

We then consider the following TVO problem We then consider the following TVO problem We then consider the following TVO problem

$$
P_1(\theta): \qquad \boldsymbol{x}^*(\theta) = \underset{\{\boldsymbol{x}_m(\theta)\}_{1}^M}{\operatorname{argmin}} \sum_{m=1}^M b_m(\theta) f(\boldsymbol{x}_m(\theta))
$$

s.t.
$$
\sum_{m=1}^M \boldsymbol{h}_m(\boldsymbol{x}_m(\theta)) = \boldsymbol{u}, \qquad (3)
$$

Instead, we develop a time-varying approach to exploit the information of optimal solution of P1(0).

Where $\mathbf{x}(\theta) = [\mathbf{x}_1^{\top}, \dots, \mathbf{x}_M^{\top}]^{\top}(\theta)$. As declared, we assume that the solution of $P_1(\theta)$ for $\theta = 0$ is given, on a step-length parameter. Where $\mathbf{x}(\theta) = [\mathbf{x}_1^\top, \dots, \mathbf{x}_M^\top]^\top(\theta)$. As declared, we assume that from low convergence rate and its solution optimality depends

And the solution at target $\theta = \tau > 0$ is to be found. Instead, we develop a time-varying

A naive approach to find the solution of $P_+(\theta)$ is to use a Newton-A have approach to find the solution of $F_1(t)$ is to use a Newton-
based iterative algorithm. However, this approach may suffer

$$
\dot{\boldsymbol{x}}_m(\theta) = \boldsymbol{\phi}_m \big(\boldsymbol{x}(\theta), \theta\big) \colon\ \mathbb{R}^{NM} \times \mathbb{R}^+ \cup \{0\} \to \mathbb{I}
$$

distance is minimized.

 \mathbb{R} is the objective function function function function function function \mathbb{R}

with optimal trajectory solution denoted by $x(\cdot)$, we then are twice commutative entirelinable with respect to $(w_1, ..., w_m)$.
devise an iterative approach to predict $x^*(\cdot)$ by $x(\cdot)$ so that the mality distance $||x(\theta) - x^*(\theta)||$ is minimized for $\theta \to \tau$. Note Assumption II: The matrices $\{\mathcal{J}_m\}_1^M$ are invertible for $\theta \in$
(A) shows a set of ODEs which should be solved with initial θ , τ , where $\mathcal{T} \in \math$ condition $x(0)$ to give the desired solution $x(\tau)$. As such, we $h_m(\cdot)$ w.r.t. $x_m(\theta)$. such TVO P₁(θ) and design { $b_m(θ) f_1$ for $θ \in$
so that the optimality distance is minimized. 3. ODES Associat With optimal trajectory solution denoted by x^* (•). We then *optimality distance* $\|\mathbf{x}(\theta) - \mathbf{x}^*(\theta)\|$ is minimized for $\theta \to \tau$. Note **Assumption II:** The matrices $\{\mathcal{J}_m\}_{m=1}^K$ condition $x(0)$ to give the desired solution $x(t)$. As such, we $n_m(\cdot)$ w.r.t. $x_m(\theta)$.
intend to jointly solve TVO $P_1(\theta)$ and design $\{b_m(\theta)\}_1^M$ for $\theta \in$ [0,*τ*] so that the optimality distance is minimized. $\frac{3}{P}$ that (4) shows a set of ODEs which should be solved with initial $[0, \tau]$, where $\mathcal{J}_m \in \mathbb{R}^{N \times N}$ is the $\frac{d}{dt}$ *b* $\frac{d}{dt}$, $\frac{d}{dt}$ $\frac{d}{dt}$, $\frac{d}{dt}$ $\frac{d}{dt}$, $\frac{d}{dt}$ $\frac{d}{dt}$ $\frac{d}{dt}$ $\frac{d}{dt}$ **with TVC** (θ) and design $\{b_m(\theta)\}_1^M$ for $\theta \in$
 $\{b_m(\theta)\}_1^M$ for $\theta \in$ n=1

Example 1. The Proposition 1. The Proposition 1. The Proposition 1. The Proposition 1. The P₁(θ),

Assumption I: The objective function function function function $\mathbf{m}(\cdot)$ are twice continuously differentiable

 $\overline{f}(\theta)$. As declared, we assume that from low convergence rate and its solution optimality depends on a step-length parameter. with a step tengui parameter. by x(·) so that the *optimality distance* ∥x(θ) − x∗(θ)∥ is minimized for θ → τ . Note that (4) shows a

Instead, we develop a time-varying approach to exploit the And the solution at target $v - v > v$ is to be found.
information of optimal solution of $P_1(0)$. We thus formulate the $fP_1(\theta)$ is to use a Newton-
this approach may suffer
this approach may suffer

$$
\dot{\boldsymbol{x}}_m(\theta) = \boldsymbol{\phi}_m\big(\boldsymbol{x}(\theta), \theta\big) : \ \mathbb{R}^{NM} \times \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^N,
$$

(•). We then are twice continuously differentiable with respect to (w.r.t.) x_m . \mathcal{A} as the objective function function function function function function \mathcal{A} ed by $x^*(\cdot)$. We then are twice continuously differentiable with respect to (w.r.t.) x_m .

Id be solved with initial $[0,\tau]$, where $\mathcal{J}_m \in \mathbb{R}^{N \times N}$ is the transpose of Jacobian matrix of $h_m(\cdot)$ w.r.t. $x_m(\theta)$.

$\frac{m}{m}$ ► ∴ $\frac{m}{m}$ ⇒ $\frac{m}{m}$ ⇒ $\frac{m}{m}$ → $\frac{m}{m}$ → , (see Fig.). The second contract of \mathcal{L} is the second contract of \mathcal{L}

 P **Proposition 1.** The solution of Karush–Kuhn–Tucker conditions **Froposition 1.** The solution of Karush-

of problem $P_1(\theta)$, for $\theta \in [0,\tau]$, can $m_{\rm eff}$

EVALUATE: Be found by the pair $(x(\theta),\lambda)$ which follows the dynamica
Assumption I: The objective function $f(\cdot)$ and constraints $h_m(\cdot)$ system (4) with: $\text{SO } P_1(\theta)$.
Be found by the pair $(x(\theta),\lambda)$ which follows the dynamical Be found by the pair $(x(\theta),\lambda)$ nts $h_m(\cdot)$ system (4) with:

$$
\phi_m(\boldsymbol{x}(\theta), \theta) = -\Big(b_m(\theta)\nabla_m^2 f + \sum_{n=1}^N \lambda_n \nabla_m^2 h_{m,n}\Big)^{-1} \mathcal{J}_m\left(\dot{\boldsymbol{\lambda}} - \frac{\dot{b}_m(\theta)}{b_m(\theta)}\boldsymbol{\lambda}\right),\tag{5}
$$

 t rre *where*

$$
\boldsymbol{\lambda} = -b_m(\theta) \boldsymbol{\mathcal{J}}_m^{-1} \nabla_m f, \qquad \text{for } m \in \{1, ..., M\},
$$
 (6)

 λ is obtained using And λ is obtained using *Proof.* Please refer to Appendix A. α is obtained using

$$
\sum_{m=1}^{M} \boldsymbol{\mathcal{J}}_m^{\top} \dot{\boldsymbol{x}}_m(\theta) = \mathbf{0}.
$$
 (7)

Proof. Please refer to Appendix A. parametric functions.

According to (5) , the dynamical system λ has been formulated has a nonlinear Combination of two perspective functions λ **P** and $b_m(\theta)$ and $\dot{b}_m(\theta)$. In the following, we use a decomposition trick We introduce the parametric functions $b_m(\theta)$ and $\dot{b}_m(\theta)$. In the following, we use a decomposition trick We introduce the parametric functions trick to express the dynamical system as a linear combination of parametric functions. According to (5), the dynamical system and occur formulated

d on a nonlinear Combination of two parametric functions, **A. Rep**

and *b*. (α) In the following we use a decomposition trick. We intr $b_m(\theta)$ and $b_m(\theta)$. In the following, we use a decomposition trick We to express the dynamical system as a linear combination of based on a nonlinear Combination of two parametric functions, \mathbf{f}

parametric functions.

A. Reparametrizing based on a Decomposition introduce the parametric functions

$$
c_m(\theta) := \frac{\dot{b}_m(\theta)}{b_m(\theta)}, \qquad m \in \{1, \dots, M\},
$$
\n(8)

should satisfy J_0 $c_m(v)dv = \log \left(\frac{v}{p_{0,m}}\right)$. *A. Reparameterizing based on a Decomposition* $\langle p_{0,m}\rangle$ *ch* should satisfy $\int_0^{\tau} c_m(\theta) d\theta = \log \left(\frac{p_{\tau,m}}{p_{0,m}} \right) := \psi_m$ bas $\overline{1}$ and \overline{r} introduce the parameteric functions \overline{r} Which should satisfy $\int_0^{\tau} c_m(\theta) d\theta = \log \left(\frac{p_{\tau,m}}{p_{0,m}} \right) := \psi_m$ based on (2). We then have: Which should satisfy $\int_0^{\tau} c_m(\theta) d\theta = \log \left(\frac{p_{\tau,m}}{n} \right) := \psi_m$ based on (2). We then have:

be re-parameterized based on the following linear combination of $\{c_m(\theta)\}_1^M$:
 $\dot{\phi}(\theta) = \phi(\phi(\theta), \theta) - \mathbf{\Gamma}(\theta), \phi(\theta)$ **Theorem 1.** The dynamical system (5) can be re-parameterized based on the following linear combination of $\{c_m(\theta)\}_1^M$:

trick to express the dynamical system as a linear combination of parameters α

 $\frac{1}{\sqrt{2}}$

 $\frac{1}{\sqrt{2}}$

γN¹ ... γNM

L

f

 $\frac{1}{\sqrt{2}}$

 $\frac{1}{\sqrt{2}}$

$$
\dot{\boldsymbol{x}}(\theta) = \boldsymbol{\phi}\big(\boldsymbol{x}(\theta), \theta\big) = \boldsymbol{\Gamma}(\theta) \, \boldsymbol{c}(\theta), \tag{9}
$$

 $vec\phi(\cdot,\cdot) = [\boldsymbol{\phi}_1^\top,\ldots,\boldsymbol{\phi}_M^\top]^\top(\cdot,\cdot),$ *where* $\boldsymbol{\phi}(\cdot,\cdot) = [\boldsymbol{\phi}_1^{\top}, \ldots, \boldsymbol{\phi}_M^{\top}]^{\top}(\cdot,\cdot),$

$$
\Gamma(\theta) = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1M} \\ \vdots & \vdots & \vdots \\ \gamma_{N1} & \cdots & \gamma_{NM} \end{pmatrix} \in \mathbb{R}^{NM \times M},
$$

\n
$$
\gamma_{nm} = -G_n^{-1} \Big(\Big(\sum_{k=1}^M D_k \Big)^{-1} D_m - \delta_{nm} I \Big) \mathbf{1},
$$

\n
$$
G_n = \text{diag}(\mathbf{v})^{-1} \mathcal{J}_n^{-1} \Big(\nabla_n^2 f + \sum_{j=1}^N v_j \nabla_n^2 h_{n,j} \Big),
$$

\n
$$
D_m = \mathcal{J}_m^{-} G_m^{-1},
$$

\n1,...,M) with
\n
$$
\mathbf{v} = -\mathcal{J}_m^{-1} \nabla_m f.
$$

\n1,...,M) with
\n
$$
\mathbf{v} = -\mathcal{J}_m^{-1} \nabla_m f.
$$

\n100
\nN matrices $\{G_n\}^M$, and exploit the following decomposition:
\n
$$
b_m(\theta) \nabla_m^2 f + \sum_{n=1}^N \lambda_n \nabla_m^2 h_{m,n} = \mathcal{J}_m \text{diag}(\mathbf{\lambda}) G_m.
$$

\n
$$
G_m = \text{diag}(\mathbf{v})^{-1} \mathcal{J}_m^{-1} \Big(\nabla_m^2 f + \sum_{n=1}^N v_n \nabla_m^2 h_{m,n} \Big),
$$

\n
$$
\theta
$$
y based on (6) and (10). Equation (12) shows that G_n is notably independent of parametric function
\n
$$
\hat{x}_n(\theta) = -G_n^{-1} \text{diag}(\mathbf{\lambda})^{-1} (\mathbf{\lambda} - c_n(\theta) \mathbf{\lambda}),
$$

\n
$$
\hat{\mathbf{\lambda}} - \mathbf{\lambda} \Big(\sum_{k=1}^M D_k \Big)^{-1} \sum_{m=1}^M D_m c_m(\theta) \mathbf{1},
$$

\n
$$
\hat{x}_n(\theta) = -G_n^{-1} \Big(\Big(\sum_{k=1}^M D_k \Big)^{-1} \sum_{m=1}^M D_m c_m(\theta) - c_n(\theta) I \Big) \
$$

 $n \in \{1,...,N\}$ and $m \in \{1,...,M\}$ with *for* n ∈ {1,...,N} *and* m ∈ {1,...,M} *with for* n ∈ {1,...,N} *and* m ∈ {1,...,M} *with for n* ∈ {1*,...,N*} *and m* ∈ {1*,...,M*} *with*

whereaver <u>van die verschieden van die verschieden van die verschieden van die verschieden van die verschieden v</u>
Die verschieden van die ver

whereaver <u>variable</u>

where ϕ(·, ·)=[ϕ[⊤]

where *where*

whereaver <u>van die ver</u>

¹ ,..., ϕ[⊤]

¹ ,..., ϕ[⊤]

¹ ,..., ϕ[⊤]

M]

M]

M]

[⊤](·, ·)*,*

[⊤](·, ·)*,*

[⊤](·, ·)*,*

$$
\boldsymbol{v} = -\boldsymbol{\mathcal{J}}_m^{-1} \nabla_m f. \tag{10}
$$

of. We introduce N-by-N matrices $\{G_m\}^M$ and exploit the following decomposition: $\frac{1}{\sqrt{1-\frac{1$ *Proof.* We introduce *N*-by-*N* matrices ${G_m}^M$ ₁^{*M*}₁ and exploit the following decomposition:

$$
b_m(\theta)\nabla_m^2 f + \sum_{n=1}^N \lambda_n \nabla_m^2 h_{m,n} = \mathcal{J}_m \operatorname{diag}(\boldsymbol{\lambda}) G_m.
$$
 (11)

1, we get: Then, we get: Then, we get: Then, we get:

$$
G_m = \text{diag}(\boldsymbol{v})^{-1} \boldsymbol{\mathcal{J}}_m^{-1} \left(\nabla_m^2 f + \sum_{n=1}^N v_n \nabla_m^2 h_{m,n} \right), \qquad (12)
$$

which we used $\lambda = b$ (θ) v based on (6) and (10). Equation (12) shows that G is notably independent of parametric function $b_n(\theta)$. Now, by plugging (11) into (5), we obtain: for which we used $\lambda = b_m(\theta)$ v based on (6) and (10). Equation (12) shows that G_m is notably independent of parametric function $b(\theta)$. Now, by plugging (11) into (5), we obtain: independent of parameters function by plugging $\mathcal{L}(\mathcal{P})$, we obtain $\mathcal{P}(\mathcal{P})$

$$
\dot{\boldsymbol{x}}_n(\theta)=-G_n^{-1}{\rm diag}(\boldsymbol{\lambda})^{-1}\big(\dot{\boldsymbol{\lambda}}-c_n(\theta)\boldsymbol{\lambda}\big),
$$

and according to (7), we get: \mathcal{L} and \mathcal{L} and \mathcal{L}

$$
\dot{\boldsymbol{\lambda}} = \boldsymbol{\lambda} \left(\sum_{k=1}^{M} D_k \right)^{-1} \sum_{m=1}^{M} D_m c_m(\theta) \mathbf{1},
$$

ch together yields: m together yields. which together yields:

$$
\dot{\boldsymbol{x}}_n(\theta)=-G_n^{-1}\bigg(\Big(\sum_{k=1}^M D_k\Big)^{-1}\sum_{m=1}^M D_m c_m(\theta)-c_n(\theta)\boldsymbol{I}\bigg)\boldsymbol{1}.
$$

Based on the definition of γ_{nm} , we thus have:

$$
\dot{\boldsymbol{x}}_n(\theta) = \sum_{m=1}^M \boldsymbol{\gamma}_{nm} c_m(\theta), \quad n \in \{1, \ldots, N\},\
$$

which proves the statement. which proves the statement. which proves the statement. which proves the statement. which proves the statement.

Remark: The dynamical system (9) depends only on *c*(•) and not on $b(\cdot)$. Moreover, as $\Gamma(\cdot)$ does not depend on $c(\cdot)$, the dynamical system portrays a linear parametric expression w.r.t. $c(\cdot)$. It enables us to find the condition in which solving (5), or

equivalently (9), leads to a unique solution.

In this regard, we make the following assumptions.

Assumption III: The matrices $\{\nabla_n^2 (f + \boldsymbol{v}^\top \boldsymbol{h}_n)\}_1^N$, $\sum_{k=1}^M D_k$ The linear form of (9) also enables us to design $c(\cdot)$ such that the and diag(v) are invertible. **Assumption III:** The matrices ${\nabla_n^2 (f + \mathbf{v}^\top \mathbf{h}_n)}_1^N$, $\sum_{k=1}^M D_k$ The linear form **In priori 111.** The matrices $\{V_n (f + v \, h_n)\}\$ $\mathcal{A}(\cdot)$ $\overline{}$, the dynamical system portrays a linear parametric expression w.r.t. c($\overline{}$ t is sumption in t , and the a depend on c(·), the dynamical system portrays a linear parametric expression w.r.t. c(·). It enables us **to find the condition in the condition in which solving (5), in the condition of the condition of the condition** to find the condition in which solving ($\frac{1}{\sqrt{2}}$), and a unique solution. In the a unique solution. In the matrices $\begin{bmatrix} v_n & v_{n+1} & v_{n+1} \end{bmatrix}$, $\angle k=1$

continuity and boundedness of the derivative, the statement follows.

 \sum_{v} optimality distance is optimized.
Assumption IV: The derivatives of $\{\nabla_m f\}_1^M$, $\{\mathcal{J}_m\}_1^M$ and Once $c(\cdot)$ is designed, we can get: $b_m(\theta) = p_{0,m} \exp\left(\int_0^{\theta} c_m(\xi) d\xi\right)$ $\{\nabla_n^2(f + \mathbf{v}^\top \mathbf{h}_n)\}_1^N$ w.r.t. $\{\mathbf{x}_m(\theta)\}_1^M$ are bounded. based on (2) and (8). **Assumption IV:** The derivatives of $\{\nabla_m f\}_1^M$, $\{\mathcal{J}_m\}_1^M$ and Once constant that

Proposition 2. If Assumptions I, II, III and IV hold, then the 4. Optimality Distance dynamical system (9) has a unique Solution. ition 2. If Assumptions 1, 11, 111 and 1V hold, then the 4. Optimality Distance
cal system (9) has a unique Solution. E Equation (9) shows a set of ODEs that is intricate to precisely

the equivalence between Lipschitz" continuity and boundedness for the derivative, the statement follows. ${\rm| d }$ continuity and bounded bounded
Second bounded bounded
 of the statement of the s

 $\sum_{k=1}^{M} D_k$ The linear form of (9) also enables us to design $c(\cdot)$ such that the $g(y)$ are invertible.
 $\frac{1}{2}$ continuity distance is optimized. \mathbf{B} $\frac{1}{2}$ diagonal diag

Proof. By utilizing the Picard–Lindelof theorem and leveraging the equivalence between Lipschitz ¨

Once $c(\cdot)$ is designed, we can get: $b_m(\theta) = p_{0,m} \exp\left(\int_0^{\theta} c_m(\xi) d\xi\right)$ based on (2) and (8) . IV. OPTIMALITY DISTANCE

hold, then the **4. Optimality Distance**

solve due to highly non-linearity w.r.t. θ . Hower
Proof. By utilizing the Picard–Lindelof theorem and leveraging is to use the Euler method to approximate $x(\cdot)$ solve due to highly non-linearity w.r.t. θ . However, one approach solve due to highly non-linearity w.r.t. θ . However, one appr
ing is to use the Euler method to approximate $x(\cdot)$ by $\hat{x}(\cdot)$: $x(x) = \frac{dy}{dx}$

$$
O_d = \|\hat{\boldsymbol{x}}(\tau) - \boldsymbol{x}(\tau)\| \le \frac{\Delta\theta^2}{2} \sum_{j=1}^L \|\ddot{\boldsymbol{x}}(\tau - j\Delta\theta)\| + \mathcal{O}(L\Delta\theta^3),
$$

where $\Delta\theta$ is the incremental step and $\hat{x}(\theta) = \phi(\hat{x}(\theta), \theta)$. For this method, the optimality distance $O_{\hat{a}}$ is upper-bounded by: α as the incremental stress where $\Delta\theta$ is the incremental step and $\hat{\mathbf{x}}(\theta) = \phi(\hat{\mathbf{x}}(\theta), \theta)$. For the where $\Delta\theta$ is the incremental step and $\hat{x}(\theta) = \phi(\hat{x}(\theta), \theta)$. For this method, the optimality distance O_d is upper-bounded by: $E_{\rm c}$ shows a set of ODEs that is intricated to precisely solve due to highly non-linearity w.r.t.

$$
\hat{\boldsymbol{x}}(\theta) = \hat{\boldsymbol{x}}(\theta - \Delta\theta) + \Delta\theta \,\hat{\boldsymbol{x}}(\theta - \Delta\theta), \qquad \theta \in (0, \tau], \tag{13}
$$

re $\tau = L\Delta\theta$. This shows that optimality distance is limited by order of $\Delta\theta$. However, based on (13) another upper-bound can be found as follows: where *t = L∆θ*. This shows that optimality distance is milited by order of ∆θ2. However, based on (13) another upper-bound can be
d as follows where $\tau = I \Delta \theta$. This shows that optimality distance is limited by order of $\Delta \theta$. However, x^{α} where $\tau = L\Delta\theta$. This shows that optimality distance is limited by order of $\Delta\theta$. However, based on (13) another upper-bound can be $\ddot{}$ $\sum_{i=1}^{n}$

$$
O_d = \left\| \int_0^{\tau} \left(\dot{x}(\theta) - \hat{m} \right) d\theta \right\| \le \int_0^{\tau} \left\| \dot{x}(\theta) - \hat{m} \right\| d\theta, \tag{14}
$$

 \mathbf{where} re where $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ are follows: where the L∆θ. This shows that optimality distance is limited by order of △ θ2. However, based on $\mathcal{L} = \mathcal{L}$ where

$$
\hat{\boldsymbol{m}} = \frac{1}{L} \sum_{j=1}^{L} \hat{\boldsymbol{x}} (\tau - j \Delta \theta).
$$

that \hat{m} does not depend on θ . Consequently, minimizing the upper bound of O_a , i.e., $\int_0^{\tau} ||\dot{x}(\theta) - \hat{m}|| \, d_\theta$, leads to the optimality that *m* does not depend on θ . Consequently, minimizing the upper b nce being minimized. \overline{c} on θ . Consequently, min: ^τ ng the upper bound of O_a , i.e., $\int_0^{\infty} ||\dot{x}(\theta) - \hat{m}||$ Note that \hat{m} does not depend on θ . Consequently, minimizing the upper bound of O_a , i.e., $\int_0^{\tau} ||\dot{x}(\theta) - \hat{m}|| d_{\theta}$, leads to the optimality $\frac{1}{4}$ distance being minimized. ⁰ ∥x˙ (θ) − 0 ∣xⁱ (θ) −

5. Optimality Distance Minimization

nsider the following functional optimization problem (FOP) to jointly design the parametric functions $c(\cdot)$ and find the un solution $x(\cdot)$: t<mark>on</mark>
nal optir We consider the following functional optimization problem (FOP) to jointly design the parametric functions $c(\cdot)$ and find the optimum solution $x(\cdot)$: optimum solution $x(\cdot)$:

EXECUTE: The probability of the binomial distribution
$$
W(t)
$$
 is a constant, $W(t)$ is a constant, $W(t)$, $W(t)$ is a constant, $W(t)$.

\nWhere $W(t)$ is a constant, $W(t)$ is a constant, $W(t)$.

\nWhere $W(t)$ is a constant, $W(t)$ is a constant, $W(t)$.

\nWhere $W(t)$ is a constant, $W(t)$ is a constant, $W(t)$.

\nTherefore, $W(t)$ is a constant, $W(t)$ is a constant, $W(t)$.

\nTherefore, $W(t)$ is a constant, $W(t)$ is a constant, $W(t)$.

\nTherefore, $W(t)$ is a constant, $W(t)$ is a constant, $W(t)$.

\nTherefore, $W(t)$ is a constant, $W(t)$ is a constant, $W(t)$

By solving FOP J_1 , we can achieve the optimal solution $x(\theta)$ which minimizes the optimality distance O_d based on (14). To solve (15), we thus constitute the Hamiltonian \mathcal{H} as:

$$
\mathcal{H} = \left\|\dot{\boldsymbol{x}}(\theta) - \hat{\boldsymbol{m}}\right\|^2 + \boldsymbol{w}(\theta)^\top \big(\dot{\boldsymbol{x}}(\theta) - \boldsymbol{\Gamma}(\theta) \, \boldsymbol{c}(\theta)\big) + \boldsymbol{\lambda}^\top \bigg(\boldsymbol{c}(\theta) - \frac{1}{\tau} \boldsymbol{\psi}\bigg),
$$

Where $w(\theta)$ is a co-state variables and λ is a Lagrange multiplier. Using calculus of variations, the Functional solution of (15) is $\frac{1}{\sqrt{15}}$ $\frac{1}{\sqrt{15}}$ where $w(\theta)$ obtained by:

 \overline{a}

$$
\begin{cases}\n\nabla_{\mathbf{c}(\theta)}\mathcal{H} = 2\mathbf{\Gamma}(\theta)^{\top} \Big(\mathbf{\Gamma}(\theta)\mathbf{c}(\theta) - \hat{\mathbf{m}} - \frac{1}{2}\mathbf{w}(\theta) \Big) + \mathbf{\lambda} = 0, \\
\nabla_{\mathbf{x}(\theta)}\mathcal{H} - \frac{d}{d\theta}\nabla_{\dot{\mathbf{x}}(\theta)}\mathcal{H} \\
= \mathbf{w}(\theta)^{\top} \nabla_{\mathbf{x}(\theta)}\mathbf{\Gamma}(\theta) \mathbf{c}(\theta) + 2\ddot{\mathbf{x}}(\theta) + \dot{\mathbf{w}}(\theta) = 0, \\
\dot{\mathbf{x}}(\theta) - \mathbf{\Gamma}(\theta) \mathbf{c}(\theta) = 0, \qquad \int_0^{\tau} \mathbf{c}(\theta) d\theta - \psi = 0.\n\end{cases}
$$
\n(16)

terative algorithm to find the solution of J_1 .

with \hat{m} being obtained ba The litions are intricate to solve and an estimation precentations are intricate to solve and an estimation precentation. These motivate us to develop an min of \hat{m} is needed in advance. These motivate us to develop an This system of conditions are intricate to solve and an estimation

functional solution of (15) is obtained by:

the beginning, we initialize $c(\cdot)$ such that the contract of the superiority of D . Contract of the set of D is D . In this regard, we propose an iterative mechanism as follows: In

If $\sum_{i=0}^{\infty}$ $\sum_{i=0}^{\infty}$ in the second step still the algorithm converges. These are the *Prediction* and Specifically, we consider the following FOP, in the second step *Furametric-tuning* steps. In the prediction step, we sequentially on OP-1 vO, to optimize the parametric functions $c(\cdot)$:
solve (13) based on (9) and recently updated $c(\cdot)$, in order to that $\int_0^{\tau} c(\theta) d\theta = \psi$. Then, we continually follow two consecutive steps till the algorithm converges. These are the *Prediction* and Specifically, we consider solve (13) based on (9) and recently updated $c(\cdot)$, in order to $\mathbf{C} = \mathbf{C} \mathbf{C} \mathbf{C} \mathbf{C} + \mathbf{C} \mathbf{C} \$ Sueps un the algorithm converges. These are the *Freutenon* and Specifically, we consider the following FOF, in the second step *Parametric-tuning* steps. In the prediction step, we sequentially of OP-TVO, to optimize the

system of conditions are intricate to solve and an estimation predict $x(\theta)$ for $\theta \in (0, \tau]$. In the parametric-tuning step, we As declared, we perform these two steps till the convergence.
Figures an iterative mechanism as follows: In We call this algorithm Optimal Parametric, Time-Varving is needed in advance. These motivate us to develop an minimize the functional objective $\int_0^{\pi} ||\dot{x}(\theta) - \hat{m}||^2 d\theta$ w.r.t. $c(\cdot)$ with \hat{m} being obtained based on the solution of prediction step.
As declared, we perform these two steps till the convergence is regard, we propose an iterative mechanism as follows: In We call this algorithm Optimal Parametric Time-Varying Optimization (OP-TVO).

of OP-TVO, to optimize the parametric functions $c(\cdot)$:

$$
J_2: \min_{\mathbf{c}(\cdot)} \int_0^{\tau} \left\| \mathbf{\Gamma}(\theta) \, \mathbf{c}(\theta) - \hat{\mathbf{m}} \right\|^2 d\theta + \mu \int_0^{\tau} \mathbf{c}(\theta) \, d\theta
$$

s.t.
$$
\int_0^{\tau} \mathbf{c}(\theta) d\theta = \psi \quad , \tag{17}
$$

First the term $\int_0^{\tau} c(\theta) \cdot \int_0^{\tau} c(\theta) d\theta$ is addition ditionally added to reg Where the term $\int_0^{\tau} c(\theta)^{\top} c(\theta) d\theta$ is additionally added to regularize the smoothness of $c(\theta)$ w.r.t. θ , and $0 < \mu \ll 1$ is the regularization coefficient. coefficient. 0 < µ ≪ 1 is the regularization coefficient.

solution of J_2 is **Proposition 3.** The globally optimal solution of J_2 is obtained by: Proposition 3. *The globally optimal solution of* J² *is obtained by:*

$$
\mathbf{c}(\theta) = \mathbf{\Pi}^{-1}(\theta) \left(\mathbf{\Gamma}(\theta)^{\top} \hat{\mathbf{m}} - \boldsymbol{\lambda} \right), \tag{18}
$$

where where where where Proposition 3. *The globally optimal solution of* J² *is obtained by:*

$$
\Pi(\theta) = \Gamma(\theta)^{\top} \Gamma(\theta) + \mu \mathbf{I},
$$

$$
\boldsymbol{\lambda} = \left(\int_0^{\tau} \Pi^{-1}(\theta) d\theta \right)^{-1} \left(\int_0^{\tau} \Pi^{-1}(\theta) \Gamma(\theta)^{\top} d\theta \hat{\mathbf{m}} - \boldsymbol{\psi} \right).
$$

pf. Considering that the dynamical system (9) has been expressed based on a linear combination of parametric functions $c(\cdot), J_2$ is *Proof.* Considering that the dynamical system (9) has been expressed based on a linear combination of parametric functions $c(\cdot)$, J_2 is a convex FOP. This implies that the globally optimal solution of J_2 can be found by applying the Euler-Lagrange equation on J_2 [17].

$$
\begin{cases}\n\Gamma(\theta)^{\top}(\Gamma(\theta) \mathbf{c}(\theta) - \hat{\mathbf{m}}) + \mu \mathbf{c}(\theta) + \lambda = 0, \\
\int_0^{\tau} \mathbf{c}(\theta) d\theta - \psi = 0.\n\end{cases}
$$
\n(19)

 \mathbf{u} c(d) = 0.000 \mathbf{u} = 0

Exigorium F shows the pseudo-code of $ST-IVO$. For each identities the value of the value of the value of $ST-IVO$. For each Algorithm 1 shows the pseudo-code of OP-TVO. For each $\sum_{d} P_{n}$ is seen a measured $\sum_{n} P_{n}$. tunity, the prediction and parametric tuning steps are **c**. Numerical results and biscussion the parametric function $x^*(\tau)$ and design the parametric functions of parametric functions of τ of τ and design the devi executed to jointly predict the optimal solution $x^*(\tau)$ and design To evaluate the devised algorithm 1 shows the parametric functions $a(\tau)$. We also need a matrix for the selection Correction Matl convergence to stop the algorithm. For this, we track the value of *Benchmark* solution obtained the parametric functions $c(\cdot)$. We also need a metric for the a Prediction-Correction Metric functions $c(\cdot)$.

$$
\hat{O}_d := \sum_{j=1}^L \left\| \phi\left(\hat{x}(\tau - j\Delta\theta), \tau - j\Delta\theta\right) - \hat{m} \right\|^2
$$
 as an estimation of
the total number of nodes of its computational complexity.
Note that, we consider this *Benchmark* as the optimal solution,
Therefore, Eefrical Electron Epg 2024

ī

the optimality distance. We thus consider that the algo
converged if the value of \hat{O}_d lies below a threshold O_{th} . \mathbb{R}^3 By solving (19), the statement follows. the optimality distance. We thus contract the primality distance. the optimality distance. We thus consider that the algorithm has

6. Numerical Results and Discussion

parametric functions $c(\cdot)$. We also need a metric for the a Prediction-Correction Method (PCM) [6], as well as with a vergence to stop the algorithm. For this we track the value of *Renchmark* solution obtained by an ext tugence to stop the argorium. For this, we track the value of *beheminal is* solution obtained by an extendity small incrementations L_{av} and $\Delta t = 10^{-6}$ recordless of its computational complexity tuning steps are extremely small incremental exercise to stop the algorithm. For this, we track the value of *Benchmark* solution obtained by an extremely small incrementaries of $\Delta\theta = 10^{-6}$ regardless of its computatio as and one of the optimality distance. We thus distance of the optimal complexity. *Benchmark* solution obtained by an extremely small incremental

by which we can compute the optimality distance O_d . These \times 1.70 GHz Intel Core i5-10310U Processor, equipped with 16 algorithms have been implemented using Matlab R2022a on a 8 GB of memory and 12 Mbytes of data cache.

Algorithm 1: Optimal Parametric Time-Varying Optimization (OP-TVO) **Algorithm 1:** Optimal Parametric Time-Varying Optimization

 $\frac{1}{\sqrt{2}}$ input: Optimal solution $x(0)$.

Outputs: Optimal solution $x(\tau)$ and parametric functions $c(\cdot)$.

$\sum_{i=1}^{\infty}$ solution $\sum_{i=1}^{\infty}$ and parameters $\sum_{i=1}^{\infty}$ and parameters c($\sum_{i=1}^{\infty}$

Initialize $c(\cdot)$ so that $\int_0^{\tau} c(\theta) d\theta = \psi$. iter = iteration = Initialize $c(\cdot)$ so

Set flag = 1 and iter = 0. $\begin{array}{ccc} & & & \searrow & \\ & & & & \searrow & \\ & & & & \searrow & \\ & & & & & \searrow & \\ & & & & & \searrow & \\ & & & & & & \searrow & \\ \end{array}$ $\sum_{i=1}^{n}$ of the $\sum_{i=1}^{n}$ \mathcal{S} and \mathcal{S}

while flag do while flag do

 $\begin{cases}\n\text{where } \text{flag} \text{ do} \\
\text{iter} = \text{iter} + 1.\n\end{cases}$ $\text{r} = \text{iter} + 1.$

Prediction step: $\mathbb{P}^1(\mathbb{P}^1 \times \mathbb{P}^1 \times \math$ Prediction step:

Predict $\mathbf{x}(\theta)$ for $\theta \in (0, \tau]$ using (13) and (9) and based on updated $\mathbf{c}(\cdot)$. Predict $\mathbf{x}(\theta)$ for $\theta \in (0, \tau]$ using (13) and (9) and based on up

Parametric-tuning step:

Update $c(\cdot)$ using (18) and based on predicted $x(\theta)$ with $\theta \in (0, \tau]$. flag = 0. if *Convergence* then Update $c(\cdot)$ using (18) and based on predicted

```
if Convergence then
                     \text{flag} = 0.end
           end
           end
if Convergence then
```

```
end
\mathcal{L} . Then constrained optimization problem E1 \mathcal{L} [18], and change the constraints to cons
```
 $\limsup_{n\to\infty} a$ constrained optimization. \blacksquare add non-linearity to the problem. We then consider a constrained optimization problem E_1 and change the constraints to add non-linearity to the problem [18.19].

$$
E_1: \qquad \min_{\{x_m\}_1^M} \sum_{m=1}^M a_m \operatorname{erfc}\left(\gamma_0 \frac{x_{m,1}}{\sqrt{2^{\frac{0.1}{x_{m,2}}}}-1}\right)
$$
\n
$$
\text{s.t.} \quad \sum_{m=1}^M \log(1+x_{m,1}) = L,
$$
\n
$$
\text{s.t.} \quad \sum_{m=1}^M x_{m,2}^2 = 1,
$$

 $\frac{1}{2}$ can be ve where the optimization variables are $x_m = [x_{m,1}, x_{m,2}]^\top$ for $m \in \{1, ..., M\}$, $M = 100$, $a_m =$ $m^{-\tau}/\sum_{m=1}^{M} m^{-\tau}$, $\tau = 3$ and $\gamma_0 = 40$. For E_1 , it can be verified that the optimal solution for $a_m = \frac{1}{M}$ is obtained as: is obtained as: is obtained as: $m^{-\tau}/\sum_{i=1}^{M} m^{-\tau}$, $\tau = 3$ and $\gamma_0 = 40$. For E_1 , it can be verified that the optimal solution $\sum_{m=1}^{n}$ is obtained as:

$$
x_{m,1} = \exp(L/M - 1), \quad x_{m,2} = \sqrt{1/M}, \quad m \in \{1, ..., M\}.
$$

we constitute a T $x = \frac{1}{\sqrt{2}}$, $x = \frac{1}{\sqrt{2}}$ Therefore, we constitute a TVO problem exactly as E₁ but with parametric functions ${b_m(\theta)}_1^M$ replacing Therefore, we constitute a TVO problem exactly as E_1 but with parametric functions $\{b_m(\theta)\}_1^M$ replacing

Figure 1: Solution Trajectories for $x_{m,1}$ with $m \in \{1,8,15\}$ Figure 2: Solution Trajectories for $x_{m,2}$ with $m \in \{20,30,100\}$

 ${a}$ $\{a_m\}_1^M$ such that: a_{11} and

$$
b_m(0) = \frac{1}{M}
$$
, $b_m(\tau) = a_m$, $m \in \{1, ..., M\}$.

We further use the re-parametrizing functions $c(\cdot)$ with Table I compares the performance resu
 $c_m(\theta) := \frac{\dot{b}_m(\theta)}{b_1(\theta)}$. We then apply Algorithm 1 with hyper-parameters and OP-TVO. The second column pre $\mu = 10^{-7}$, $O_{\text{th}} = 10^{-5}$ and $\Delta\theta = 10^{-2}$ to find the optimal solution.
extent of constraints violation, represented as We further use the re-parametrizing functions $c(•)$ with Table I comp $\frac{b_m(\theta)}{b_m(\theta)}$. We then apply Algorithm 1 with hyper-parameters

Figures 1 and 2 illustrate the solution trajectories $\{x_m(\theta)\}_1^M$, of violations of all constraints. The fourth column in portray a highly nonlinear behavior. However, as the number O_a , respectively.

of iteration increases, this non-linearity reduces. When the optimality distance has been obtained. So ignificantly in terms of iterations. For the first iteration, the trajectory solutions sixth columns show the optimality distance O_d and its of iteration increases, this non-linearity reduces. When the algorithm converges (3rd iteration), the linear curvature of the obtained by Algorithm 1, as a function of θ for different elapsed time in seconds fo solution trajectories indicate that a precise solution with minimal value than Benchmark, it is $Fig. 12.12$

have a fair comparison, we adjust the hyper-parameter $\Delta\theta$ for
RCM such that the compared in a subject is almost a such to that of Algorithm 1. As such, we need to set $\Delta\theta = 10^{-4}$. PCM such that the corresponding optimal value is almost equal T_{max} value is allowed equal.

pares the perfo second and OP-TVO. The second column presents the optimal values $\frac{M}{1}$, of violations of all constraints. The fourth column indicates the tions. For the first iteration, the trajectory solutions sixth columns show the optimality distance O_d and its estimation Table I compares the performance results of PCM, Benchmark extent of constraints violation, represented as the summation achieved by these approaches. The third column quantifies the elapsed time in seconds for the computations. And the fifth and \hat{O}_d , respectively.

rithm converges (3rd iteration), the linear curvature of the Despite PCM achieving a slightly lower objective function We also apply a PCM on Problem E_1 to obtain the solution. To reference in this comparison. Based on the values of \hat{O}_a , imality distance has been obtained. The solution of significantly in terms of constraint satisfaction compared to value than Benchmark, it is noteworthy that Benchmark excels PCM. Therefore, the Benchmark solution is considered the reference in this comparison. Based on the values of \hat{O} . reference in this comparison. Based on the values of O_{d} ,

Approach		Optimal Value Constraint Violations Elapsed Time [s]		O_d	\hat{O}_d
Benchmark	5.6089×10^{-7}	1.312×10^{-6}	2066	0.0	N/A
PCM	5.6086×10^{-7}	1.739×10^{-4}	271	0.00135	N/A
$OP-TVO$, iter=1	3.2443×10^{-5}	0.419	3	0.8291	N/A
$iter=2$	5.6088×10^{-7}	5.129×10^{-5}	52	5.132×10^{-4}	0.228
$iter=3$	5.6088×10^{-7}	5.051×10^{-5}	105	5.130×10^{-4}	2.68×10^{-6}
$iter=4$	5.6088×10^{-7}	5.050×10^{-5}	160	5.130×10^{-4} 2.14×10^{-6}	

Table 1: Performance Result of OP-TVO and PCM

OP-TVO chieves convergence after three iterations. OP-TVO with iter $= 3$ outperforms PCM from the computational complexity as it converges within $10⁵$ seconds while PCM converges after 271 seconds. Furthermore, OP-TVO with iter = 3 exhibits a superior optimality distance O_d compared to PCM. Not to mention that OP-TVO better satisfies the constraints than PCM. These results indicate that OP-TVO provides a more accurate solution than PCM, demonstrating its capability to achieve optimal solutions with lower computational complexity and higher performance precision.

7. Conclusion

In this paper, we reformulated a class of nonlinear constrained optimization problems using a time varying optimization approach incorporating parametric functions. By applying a reparametrization technique, we transformed the problem into a dynamical system expressed linearly in terms of these parametric functions. Our objective was to minimize the optimality distance traced by this dynamical system, which led us to formulate a functional minimization problem. To achieve this, we introduced an iterative algorithm named OP-TVO, designed specifically to determine the trajectory of solutions with an optimal optimality distance. Our experimental results demonstrate that OP-TVO surpasses the Prediction-Correction Method (PCM) in terms of both optimality distance and convergence rate. These findings highlight OP-TVO as a promising alternative to PCM for addressing distributed time-varying optimization problems. By building upon the results gained from this study, future work can explore optimization problems involving time-varying constraints.

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A. Proof of Proposition 1. **Appendix**

A. Proof of Proposition 1. **A. Proof of Proposition 1.**

stitute the Lagrangian function, and obtain the Karush–Kuhn–Tucker conditions which read: A. Proot of Proposition 1.
We constitute the Lagrangian function, and obtain the Karush–Kuhn–Tucker conditions which read:

$$
\begin{cases}\n b_m(\theta)\nabla_m f + \mathcal{J}_m \lambda = 0, \\
 \sum_{m=1}^M \mathbf{h}_m(\mathbf{x}_m) - \mathbf{u} = 0,\n\end{cases}
$$
\n(20)

 κ l is the Lagrange multiplier. By considering Assumption 1 and taking derivative of the recent conditions wet θ we get where λ is the Lagrange multiplier. By considering Assumption 1 and taking derivative of the recent conditions w.r.t. θ , we get:

<u>APPENDIX</u>

$$
\begin{cases}\n\dot{b}_{m}(\theta)\nabla_{m}f + \left(b_{m}(\theta)\nabla_{m}^{2}f + \sum_{j=1}^{N}\lambda_{j}\nabla_{m}^{2}h_{m,j}\right)\dot{x}_{n}(\theta) \\
+ \mathcal{J}_{m}\dot{\lambda} = 0, \\
\sum_{m=1}^{M}\mathcal{J}_{m}^{\top}\dot{x}_{m} = 0,\n\end{cases}
$$
\n(21)

By combining (20) and (21) and considering Assumption 2, the statement follows. By combining (20) and (21) and considering **Assumption 2**, the statement follows.

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