ISSN: 2834-7706

Research Article

Journal of Mathematical Techniques and Computational Mathematics

A Finite Difference Scheme for Smooth Solutions of the General mKDV Equation

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Submitted: 2024, Jun 27; **Accepted:** 2024, Jul 15; **Published:** 2024, Aug 05

Citation: Rodriguez, J. N., Omel'yanov, G. (2024). A Finite Difference Scheme for Smooth Solutions of the General mKDV Equation. *J Math Techniques Comput Math, 3*(8), 01-15.

Abstract
We consider a generalization of the mKdV model of shallow water out-flows. This generalization is a family of equations with nonlinear dispersion terms containing, in particular, KdV, mKdV, Benjamin- Bona-Mahony, Camassa-Holm, and Degasperis-Procesi equations. Non- linear dispersion, generally speaking, implies instability of classical solutions and wave breaking in a finite time. However, there are special conditions under which the general mKdV equation admits classical solutions that are global in time. We have created an economic finite difference scheme that preserves this property for numerical solutions. To illustrate this we demonstrate some numerical results about propogation and interaction of solitons.

Keywords: General mKdV Equation, Degasperis-Procesi Model, Finite Difference Scheme, Soliton, Interaction
2010 Mathematics Subject Classification: 35Q35, 35Q53, 65M06

1. Introduction

We consider a modern unidirectional approximation of the shallow water system (see e.g, [1]) called the "general mKdV equation" (gmKdV):

$$\frac{\partial}{\partial t}\left\{u-\alpha^2\varepsilon^2\frac{\partial^2 u}{\partial x^2}\right\} \tag{1}$$

$$+\frac{\partial}{\partial x}\left\{c_0 u+c_1 u^n-c_2\varepsilon^2\Big(\frac{\partial u}{\partial x}\Big)^2+\varepsilon^2\big(\gamma-c_3 u\big)\frac{\partial^2 u}{\partial x^2}\right\}=0,\ x\in\mathbb{R}^1,\ t>0.$$

Here $\alpha, c_0, \ldots, c_3, \gamma$ are real parameters and ε characterizes the dispersion, n = 2 or n = 3. The constants $\alpha \geq 0$ and $\gamma \geq 0$ are associated with different characters of the dispersion manifestation. In the Green-Naghdi approximation the restriction $\alpha + \gamma > 0$ is required. The equation (1) terms with $c_2 \geq 0$ and $c_3 \geq 0$ can be treated as representations of nonlinear dispersion. In the Camassa-Holm approximation $c_2 + c_3 > 0$.

This six parametric family of third order conservation laws contains as particular cases a list of basic equations: the KdV and mKdV equation if $\alpha = c_2 = c_3 = 0$ and $n = 2$ or $n = 3$; the Benjamin-Bona-Mahony (BBM) equation ([2], 1972) if $n = 2$, $\gamma = c_2 = c_3 = 0$; the Camassa-Holm (CH) equation ([3], 1993) if $n = 2$, $c_2 = c_3 / 2$, $c_1 = 3c_3/2\alpha^2$, and $\gamma = 0$; and the Degasperis-Procesi (DP) equation ([1], see also [4]) if $n = 2$, $c_2 = c_3$, $c_1 = 2c_3/\alpha^2$, and $c_0 = \gamma = 0$. All these particular equations are quite different. Indeed, the KdV, BBM and generalizations of CH and DP (if $\gamma + \alpha^2 c_0 > 0$) equations have soliton-type traveling wave solutions. At the same time, the CH with $c_0 = 0$ and DP equations, under the condition $u \to 0$ as $x \to \pm\infty$, have non-smooth traveling wave solutions only. Next, the KdV, mKdV, CH, and DP equations are completely integrable, whereas all others particular cases of the model (1) are essentially non-integrable (see e.g. [5]). Consequently, KdV, mKdV, DP, and CH solitons collide elastically, whereas BBM "solitons" have changed after the interaction and an oscillatory tail is generated [6]. Furthermore, the Cauchy and periodic problems for the CH and DP equations have been studied extensively (see e.g. [4,5,7,8] and references therein), whereas the solvability of similar problems for the general case (1) remains be unknown. So, the general model (1) represents the non trivial object of investigations and it is naturally to expect a very interesting behavior of its solution.

Concerning the numerical modeling of the equations of family (1), it seems that quite a lot of researches has been carried out. The main part of them is devoted to the CH equation. The fact is that when $c_3 = 2c_2$, the main term that generates instability falls out of the main balance law

$$\frac{d}{dt}\left\{\int_{-\infty}^{\infty} u^2 dx + \alpha^2 \int_{-\infty}^{\infty} (\varepsilon u_x)^2 dx\right\} = \varepsilon^{-1}(c_3 - 2c_2)\int_{-\infty}^{\infty} (\varepsilon u_x)^3 dx. \qquad (2)$$

Further, due to wave breaking and the existence of non-smooth solutions, the proposed schemes for CH and DP equations are mainly based on Fourier representation or Galerkin method (e.g. [9,-12] and others). Obviously, this involves the use of fairly complex iterative procedures.

As for the finite difference approximation, the approach that is mainly developed here is one that uses bi-Hamiltonian structures of CH and DP equations - the so-called "mean vector field method" [13-18].

Our interest in the numerical investigation of the gDP model (for $n = 2$) appeared after the first result which states that gDP solitons, under some conditions, interact elastically in the weak asymptotic (for $\varepsilon \to 0$) sense [19,20]. To demonstrate this effect numerically there has been developed an approach which adapts to the finite-difference representation the conservation law

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\, dx = 0, \qquad (3)$$

and the balance law (2). Next, let us note that right-hand side in (2) disappears for even functions. Thus, numerical simulation of soliton motion seems to be stable. Subsequent numerical experiments showed the validity of this hypothesis. Moreover, it turned out that the collision of solitons, in which, generally speaking, the evenness of the solution is violated, does not lead to any significant errors.

In this paper we adapt the main ideas of [23] and previous articles [21,22] to the general mKd model (1) for n = 3 and create a "conservative" and effective algorithm. The content of the paper is the following: in Section 2 we present assumptions which guarantee the soliton solution existence, a description of the finite difference scheme for sufficiently smooth waves is contained in Section 3, which focuses on the problem of dynamics and interaction of solitons. Section 4 shows the results of the corresponding numerical experiments. In the appendix, we demonstrate some technical details of the finite difference scheme analysis.

2. Soliton Type Solution

The difference between gDP and gmKdV equations is much deeper than between KdV and mKdV. The most important novelty here is such that solitons and antisolitons have different shapes. Furthermore, both solitons and antisolitons move with positive velocities.

To clarify the method for constructing a smooth traveling wave for equation (1), we briefly explain the approach [24]. Let's consider the ansatz

$$u = A\omega\Big(\beta(x - Vt - x^0)/\varepsilon, A\Big), \qquad (4)$$

which we call soliton (for $A > 0$ or antisoliton for $A < 0$), regardless of the scenario of their interaction. Here $\omega(\eta, A)$ is a smooth function such that

$$\omega(0, A) = 1, \quad \omega(-\eta, A) = \omega(\eta, A), \quad \omega(\eta, A) \to 0 \quad \text{as} \quad \eta \to \pm\infty, \qquad (5)$$

the wave amplitude $A \neq 0$, the scale β, and the initial point x^0 are free parameters. The velocity $V = V(A) \neq 0$ should be determined. We assume

$$\gamma + \alpha^2 V > 0, \qquad (6)$$

and, to simplify subsequent formulas, define the notation

$$\begin{aligned} &r = c_3/(c_2 + c_3),\ \beta = \sqrt{c_1(\gamma + \alpha^2 V)}/(c_3\sqrt{r}), \\ &q = c_3^2(V - c_0)/\Big(c_1(\gamma + \alpha^2 V)^2\Big),\ p = c_3 A/(\gamma + \alpha^2 V). \end{aligned} \qquad (7)$$

Next we define a new function $g = g(\eta, q)$ such that

$$\omega(\eta, A) = \Big(1 - g(\eta, q)^r\Big)/p. \qquad (8)$$

Substituting (8) into (1) we pass to the equation

$$\left(\frac{dg}{d\eta}\right)^2 = F(g,q), \tag{9}$$

Where

$$\begin{aligned} F(g,q) &= 3g^2 - 2\frac{1}{2+r}g^{2+r} - 2\frac{3-q}{2-r}g^{2-r} + \frac{1-q}{1-r}g^{2-2r} - C(q), \\ C(q) &= \frac{r\big(3r^2 - q(2+r)\big)}{(1-r)(4-r^2)}. \end{aligned} \tag{10}$$

The right-hand side F has three roots. One of them, $g = 1$, corresponds to the behavior of ω at infinity. The other, $g_0 < 1$ and $g_1 > 1$, are associated with the condition $\omega(0, A) = 1$. It's easy to verify that

$$g_0 \in (0,1) \iff C(q) > 0, \quad F''_{gg}\Big|_{g=1} > 0 \iff q > 0. \tag{11}$$

Assuming the fulfillment of the assumptions (11) we obtain the function *F (g, q)* like depicted on Figure 1.

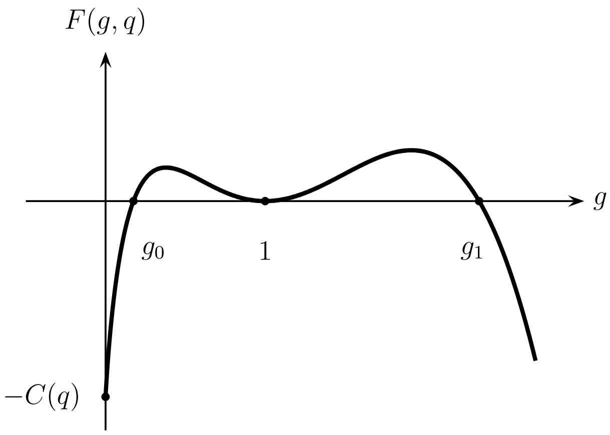

Figure 1: Right-Hand Side of the Equation (9) in the case $r = 1/2$, $q \approx 0.148$. Here $g_0 \approx 0.175$, $g_1 \approx 2.455$, and $C(q) \approx 1.964$.

Now we note that, for the representation (8), the equality

$$1 - g_k^r \stackrel{\text{def}}{=} p_k = c_3 A/(\gamma + \alpha^2 V), \quad k = 0, 1, \tag{12}$$

can be realized if and only if $k = 0$ for $A > 0$ and $k = 1$ for $A < 0$. In turn, equalities (12) allow us to determine the relationship between the speed $V = V_k$ and root g_k. For $\alpha > 0$ we have

$$V_k = \alpha^{-2}\Big\{c_3 A/p_k - \gamma\Big\}, \quad k = 0, 1. \tag{13}$$

Therefore, we obtain the equations for the roots $g_k = g_k(A)$

$$F(g_k, q)\Big|_{q=q(V_k)} = 0, \quad k = 0, 1. \tag{14}$$

Now we can state the Cauchy problems

$$\frac{dg}{d\eta} = \sqrt{F(g,q)}\Big|_{q=q(V_0)}, \quad \eta \in (0,\infty); \quad g|_{\eta=0} = g_o, \tag{15}$$

if $A > 0$ and

$$\frac{dg}{d\eta} = -\sqrt{F(g,q)}\Big|_{q=q(V_1)}, \quad \eta \in (0,\infty); \quad g|_{\eta=0} = g_1, \tag{16}$$

if $A < 0$

Considering in the same manner the case $\alpha = 0$, we find the root $g_0 = \overline{g}_0(A)$ of F and the associated velocity $\overline{V}$

$$\bar{g_0}(A) = (1 - c_3 A/\gamma)^{1/r}, \quad \bar{V} = c_0 + c_1\gamma^2 q\Big(\bar{g_0}(A)\Big)/c_3^2, \quad \alpha = 0. \tag{17}$$

Thus, taking into account the restrictions (11), we come to the following statement.

Theorem 1

Let the amplitude A satisfy the conditions [24]

$$A \in (A_0^*, A_0^-)\bigcup(A_0^+, \infty) \quad \textit{if} \quad \alpha > 0, \quad \gamma_\alpha > 0, \quad \textit{and} \quad c_3^2 > 4\xi\gamma_\alpha, \tag{18}$$

$$A > A_0^*, \quad A \neq \overline{A_0^\pm} \quad \textit{if} \quad \alpha > 0, \quad \gamma_\alpha > 0, \quad \textit{and} \quad c_3^2 = 4\xi\gamma_\alpha, \tag{19}$$

$$A > A_0^* \quad \textit{if} \quad \alpha > 0, \quad \gamma_\alpha > 0, \quad \textit{and} \quad c_3^2 < 4\xi\gamma_\alpha, \tag{20}$$

$$A < p_1\gamma_\alpha c_3 < 0 \quad \textit{if} \quad \alpha > 0, \tag{21}$$

either $A < \gamma/C_3$, $A \neq 0$ *if* $\alpha = 0$. *Then the equation (1) for* $n = 3$ *has the soliton solution (4) with the velocity (13) if* $\alpha > 0$ *and (17) if* $\alpha = 0$. *The function* $\omega(\eta, A)$ *vanishes at the exponential rate,* $\omega(\eta, A) \sim \exp(-\sqrt{rq}\eta)$ *for* $\eta >> 1$. *Here* $\gamma_\alpha = \gamma + \alpha^2 c_0$, $\xi = 3r^2\alpha^2 c_1/(2 + r)$, $A_0^* = p_0\gamma_\alpha/c_3$,

$$\overline{A_0^\pm} = p_0 c_3/2\xi, \ \textit{and} \ A_0^\pm = p_0\Big(c_3 \pm \sqrt{c_3^2 - 4\xi\gamma_\alpha}\Big)/2\xi.$$

3. Finite Difference Scheme

The actual numerical simulation for the Cauchy problem for the gmKdV equation (1) is realized for $n = 3$ and for a bounded x-interval, $x \in [0, L]$. For this reason we simulate the Cauchy problem by the following mixed problem:

$$\frac{\partial}{\partial t}\Big\{u - \alpha^2\varepsilon^2\frac{\partial^2 u}{\partial x^2}\Big\} + \frac{\partial}{\partial x}\Big\{c_0 u + c_1 u^3 - c_2\varepsilon^2\Big(\frac{\partial u}{\partial x}\Big)^2 + \gamma\varepsilon^2\frac{\partial^2 u}{\partial x^2} - c_3\varepsilon^2 u\frac{\partial^2 u}{\partial x^2}\Big\} = 0, \quad x \in (0, L), \quad t \in (0, T), \tag{22}$$

$$u\Big|_{x=0} = u\Big|_{x=L} = u_x\Big|_{x=L} = 0, \quad u\Big|_{t=0} = u^0(x/\varepsilon), \tag{23}$$

where L, T, and sufficiently smooth function u^0 are such that uniformly in $t \in [0, T]$

$$\Big|u(x,t)|_{x\in[0,\delta]}\Big| \leq c\varepsilon^2 << 1, \quad \Big|u(x,t)|_{x\in[L-\delta,L]}\Big| \leq c\varepsilon^2 << 1 \tag{24}$$

for a sufficiently small $\delta > 0$.

When modeling interaction phenomena according to (22), (23), we take into account that explicit formulas for multisoliton solutions to the equation (1) remain unknown. However, solitons are functions that decay with exponentia rates, so a linear combination of solitary waves (4) approximates the exact multisoliton solution if the distances between the waves are large enough. In particular, to approximate the two-soliton solution we set

$$u^0 = \sum_{i=1}^{2} A_i\omega\Big(\beta_i(x - x_i^0)/\varepsilon, A_i\Big), \tag{25}$$

and assume that

$$\frac{1}{\varepsilon}|x_2^0 - x_1^0| \geq c\varepsilon^{-\mu} >> 1, \quad \mu > 0. \tag{26}$$

Accordingly, in the case of (25) the parameters L, T, ε and the initial positions of the fronts x_i^0 should be such that the intersection points of the trajectories of solitary waves belong to the domain $Q_T = (0, L) \times (0, T)$.

Obviously, it is impossible to create any finite difference scheme for a problem with singular perturbations which remains reasonable uniformly in $\varepsilon \to 0$ and $t \in (0, T)$, $T =$ const. So we treat ε as a small but fixed constant. However, we fix any relation between ε and finite difference scheme parameters.

To create a finite differences scheme for the equation (22) we should choose appropriate approximations for the differential and nonlinear terms. Let us do it separately.

3.1 Preliminary Nonlinear Scheme

As usually, we define a mesh $Q_{T,\tau,h} = \{(x_i, t_j) \stackrel{\text{def}}{=} (ih, j\tau), i = 0, \dots, I,$ $j = 0, \dots, J\}$ over Q_T and denote

$$y_i^j \stackrel{\text{def}}{=} u(x_i, t_j), \quad y_{ix}^j \stackrel{\text{def}}{=} \partial_x y_i^j \stackrel{\text{def}}{=} \frac{y_{i+1}^j - y_i^j}{h}, \quad y_{i\bar{x}}^j \stackrel{\text{def}}{=} \partial_{\bar{x}} y_i^j \stackrel{\text{def}}{=} \frac{y_i^j - y_{i-1}^j}{h},$$

$$y_{i\mathring{x}}^j \stackrel{\text{def}}{=} \partial_{\mathring{x}} y_i^j \stackrel{\text{def}}{=} \frac{1}{2}(y_{ix}^j + y_{i\bar{x}}^j), \quad y_{i\bar{t}}^j \stackrel{\text{def}}{=} \partial_{\bar{t}} y_i^j \stackrel{\text{def}}{=} \frac{y_i^j - y_i^{j-1}}{\tau}, \quad y_{ix\bar{x}}^j = (y_{ix}^j)_{\bar{x}}.$$

Let us consider the following system of nonlinear equations:

$$y_{i\bar{t}}^j - \alpha^2\varepsilon^2 y_{ix\bar{x}\bar{t}}^j + c_0 y_{i\mathring{x}}^j + c_1 Q_1(y_i^j) + \varepsilon^2\Big(\gamma_h y_{ix\bar{x}\mathring{x}}^j + \gamma h y_{ix\bar{x}x}^j\Big)$$

$$- \varepsilon^2 Q_2(y_i^j) = 0, \quad i = 1, \dots, I-1, \quad j = 1, 2, \dots, J, \quad (27)$$

$$y_l^j = 0, \quad y_{I-l}^j = 0, \quad l = 0, 1, 2, \quad j = 1, 2, \dots, J, \quad (28)$$

$$y_i^0 = \widetilde{u}^0(x_i/\varepsilon), \quad i = 0, \dots, I, \quad (29)$$

where

$$Q_1(y) = \frac{1}{2}\{y^2 y_{\mathring{x}} + y(y^2)_{\mathring{x}} + (y^3)_{\mathring{x}}\}, \quad \gamma_h = \gamma(1-h), \quad (30)$$

$$Q_2(y) = \partial_{\mathring{x}}\Big\{c_2 y_x y_{\bar{x}} + \frac{c_3}{2}\Big(2y y_{x\bar{x}} + (y_x)^2 - 2y_x y_{\bar{x}} + (y_{\bar{x}})^2\Big)\Big\}, \quad (31)$$

$$\widetilde{u}^0(x_l/\varepsilon) = \widetilde{u}^0(x_{I-l}/\varepsilon) = 0 \quad \text{for} \quad l = 0, 1, 2,$$

$$\widetilde{u}^0(x_i/\varepsilon) = \frac{1}{h}\int_{x_i - h/2}^{x_i + h/2} u^0\left(\frac{\eta}{\varepsilon}\right) d\eta, \quad i = 3, \dots, I-3.$$

The main properties of the terms $Q_l(y)$ are the following

$$h\sum_i Q_l(y_i) = 0 \quad l = 1, 2, \quad h\sum_i y_i Q_1(y_i) = 0, \quad (32)$$

$$h\sum_i y_i Q_2(y_i) = \frac{1}{2}(c_3 - 2c_2) h \sum_i y_{ix} y_{i\bar{x}} y_{i\mathring{x}}. \quad (33)$$

The sense of the term $hy_{ix\bar{x}x}^j$ is similar to the parabolic regularization of the gDP equation (see below). It is clear also that the local approximation accuracy of (27) is $O(\tau+h^2)$ for sufficiently smooth solution. It should be noted also that a similar approach to the nonlinearity Q_l digitization has been presented and successfully used in [25,26].

To simplify the notation, we write

$$y \stackrel{\text{def}}{=} y_i^j, \quad \check{y} \stackrel{\text{def}}{=} y_i^{j-1}.$$

So, the short form of the equation (27) is the following:

$$y_{\bar{t}} - \alpha^2\varepsilon^2 y_{x\bar{x}\bar{t}} + c_0 y_{\mathring{x}} + c_1 Q_1(y) + \varepsilon^2\Big(\gamma_h y_{x\bar{x}\mathring{x}} + \gamma h y_{x\bar{x}x}\Big) - \varepsilon^2 Q_2(y) = 0. \quad (34)$$

Our first result consists of obtaining of discrete analogs of the equalities (2) and (3). Multiplying (27) by 1 and *y*, summing over *i*, and using some trivial equalities (see [23]) it is easy to obtain the following

Lemma 1

Let the nonlinear system of algebraic equations (27)-(29) have the unique solution $y_i^j,\ i = 1, \dots, I-1,\ j = 1, 2, \dots, J,$ and let $\varepsilon > 0$ be a constant. Then uniformly in $j \le J$ the following relations hold:

$$\partial_{\bar t} h \sum y^j = 0, \tag{35}$$

$$\partial_{\bar t}\Big\{\|y^j\|^2 + \alpha^2\|\varepsilon y^j_x\|^2\Big\} + \tau\Big\{\|y^j_{\bar t}\|^2 + \alpha^2\|\varepsilon y^j_{x\bar t}\|^2\Big\} + \gamma h^2\|\varepsilon y^j_{x\bar x}\|^2$$
$$= \varepsilon^2(c_3 - 2c_2)h\sum y^j_x y^j_{\bar x} y^j_{\mathring{x}}, \tag{36}$$

Here and in what follows $\sum$ denotes the summation over all i and $\|\cdot\|$ is the discrete version of the $L^2(0, L)$ norm, namely

$$\sum f = \sum_{i=1}^{I-1} f_i\,, \quad \|f\|^2 = h\sum_{i=1}^{I-1}|f_i|^2. \tag{37}$$

Next, we note that the equality (36), as well as (2), does't imply any regu- larity of the solution if $c_3 \neq 2c_2$. However, there exists a special case, when (36) allows us to analyze the equation (34) solution.

Lemma 2

Let $T > 0$ be a constant, $\alpha > 0$, and u^0, $u^0_x \in L^2(0, L)$. Moreover, let for each $j = 0, 1, 2, ..., J$ there exist $k(j)$ such that

$$y^j_{k(j)+i} = y^j_{k(j)-i}, \quad i = 0, 1, \ldots, \tag{38}$$

where $y^j_i = 0$ *for* $i \le 0$ *and* $i \ge I$. *Then under the assumptions of Lemma 1*

$$\|y^j\|^2 + \alpha^2\|\varepsilon y^j_x\|^2 + \tau\Big\{|||y_{\bar t}|||^2(j) + \alpha^2|||\varepsilon y^j_{x\bar t}|||^2(j)\Big\}$$
$$+\gamma h^2|||\varepsilon y_{x\bar x}|||^2(j) = \|y^0\|^2 + \alpha^2\|\varepsilon y^0_x\|^2. \tag{39}$$

Here and in what follows $|||\cdot|||(j)$ is the discrete version of the $L^2\Big((0, L)\times(0, t_j)\Big)$ norm, namely

$$|||f|||^2(j) = \tau\sum_{k=1}^{j}\|f^k\|^2. \tag{40}$$

As a consequence of this lemma one can prove a convergence result. Namely, let $y_{\tau,h}(x,t)$ be an extension of the net-function $y_i^{\,j}$ which satisfies the same estimate (39) and the evenness assumption in the sense of (38) (see e.g. [27]), and let $u(x, t)$ denote the solution of the problem (22), (23). Then, using the standard technic (see e.g. [28]), one can prove the theorem

Theorem 2

Let the assumptions of Lemma 2 be satisfied. Then there exists a subsequence $y_{\bar\tau,\bar h}(x, t)$ such that

$$y_{\bar\tau,\bar h} \to u \quad *\text{-weakly in} \quad L^\infty\Big((0, T); W^1_2(0, L)\Big) \tag{41}$$

as τ, $h \to 0$, where W^1_2 *denotes the Sobolev space.*

Similar to (39) one can obtain stronger estimates.

Lemma 3. Under the assumption of Lemma 2 let $u^0 \in W^2_2$. Then uniformly in j:

$$\|\varepsilon y^j_x\|^2 + \alpha^2\|\varepsilon^2 y^j_{x\bar x}\|^2 + \tau\Big\{|||\varepsilon y_{x\bar t}|||^2(j) + \alpha^2|||\varepsilon^2 y_{x\bar x\bar t}|||^2(j)\Big\}$$
$$+\gamma h^2|||\varepsilon^2 y_{x\bar x\mathring{x}}|||^2(j) = \|\varepsilon y^0_x\|^2 + \alpha^2\|\varepsilon^2 y^0_{x\bar x}\|^2, \tag{42}$$

$$\|y^j_{\bar t}\|^2 + \|\varepsilon y^j_{x\bar t}\|^2 \le \varepsilon^{-4}C\Big(\|y^0\|, \|\varepsilon y^0_x\|, \|\varepsilon^2 y^0_{x\bar x}\|\Big), \tag{43}$$

where $C(v, w, z)$ does not depend on τ, h, and ε.

3.2 Linearization

Now we should verify the solvability of the equation (34) for any fixed $j \ge 1$, that is of the system of nonlinear equations

$$y-\alpha^2\varepsilon^2 y_{x\bar{x}}+\tau\Big\{c_0 y_{\dot{x}}+c_1 Q_1(y)+\varepsilon^2\Big(\gamma_h y_{x\bar{x}\dot{x}}+\gamma h y_{x\bar{x}x}\Big)$$
$$-\varepsilon^2 Q_2(y)\Big\}=\check{y}-\alpha^2\varepsilon^2\check{y}_{x\bar{x}}, \qquad (44)$$

as well as select a way to linearize the nonlinear terms. To this aim let us construct the sequence of vector functions $\varphi(s)\overset{\text{def}}{=}\{\varphi_0(s),...,\varphi_I(s)\}$, $s\geq 0$, such that $\varphi(0)=\check{y}$ and $\varphi\overset{\text{def}}{=}\varphi(s)$ for $s\geq 1$ satisfies the equation

$$\begin{aligned}&\varphi-\alpha^2\varepsilon^2\varphi_{x\bar{x}}+\tau\Big\{c_0\varphi_{\dot{x}}+c_1R_1(\bar{\varphi},\varphi)+\varepsilon^2\Big(\gamma_h\varphi_{x\bar{x}\dot{x}}+\gamma h\varphi_{x\bar{x}x}\Big)\\&-\varepsilon^2R_2(\bar{\varphi},\varphi)\Big\}=\check{y}-\alpha^2\varepsilon^2\check{y}_{x\bar{x}}+\tau\Big\{2c_1Q_1(\bar{\varphi})-\varepsilon^2Q_2(\bar{\varphi})\Big\}, \qquad (45)\\&\varphi_l=0,\quad \varphi_{I-l}=0,\quad l=0,1,2,\end{aligned}$$

where $\bar{\varphi}\overset{\text{def}}{=}\varphi(s-1)$. To obtain (45) we use the identity

$$Q_1(\varphi)=Q_1(\bar{\varphi}+\mathrm{w})=Q_1(\bar{\varphi})+R_1(\bar{\varphi},\mathrm{w})+R_1(\mathrm{w},\bar{\varphi})+Q_1(\mathrm{w}), \qquad (46)$$

$$Q_2(\varphi)=Q_2(\bar{\varphi}+\mathrm{w})=Q_2(\bar{\varphi})+R_2(\bar{\varphi},\mathrm{w})+Q_2(\mathrm{w}), \qquad (47)$$

$$R_1(u,v)=\frac{1}{2}\Big\{u^2v_{\dot{x}}+2u(uv)_{\dot{x}}+3(u^2v)_{\dot{x}}+2uvu_{\dot{x}}+v(u^2)_{\dot{x}}\Big\}, \qquad (48)$$

$$\begin{aligned}R_2(\bar{\varphi},\mathrm{w})=\partial_{\dot{x}}\Big\{&(c_2-c_3)\Big(\bar{\varphi}_x\mathrm{w}_{\bar{x}}+\bar{\varphi}_{\bar{x}}\mathrm{w}_x\Big)\\&+c_3\Big(\bar{\varphi}\mathrm{w}_{x\bar{x}}+\bar{\varphi}_x\mathrm{w}_x+\bar{\varphi}_{\bar{x}}\mathrm{w}_{\bar{x}}+\bar{\varphi}_{x\bar{x}}\mathrm{w}\Big)\Big\}, \qquad (49)\end{aligned}$$

where $w=\varphi-\bar{\varphi}$. Next we neglect the quadratic in w terms in (46), (47), and note that

$$R_1(\bar{\varphi},\mathrm{w})=R_1(\bar{\varphi},\varphi)-3Q_1(\bar{\varphi}),\quad R_2(\bar{\varphi},\mathrm{w})=R_2(\bar{\varphi},\varphi)-2Q_2(\bar{\varphi}). \qquad (50)$$

Continuing, we note that the solvability of the algebraic system (45) is obvious for sufficiently small τ/h^3. In order to estimate $\|\varphi\|$ let us use the identities (46) - (50) again and rewrite (45) as follows:

$$\begin{aligned}&\varphi-\alpha^2\varepsilon^2\varphi_{x\bar{x}}+\tau\Big\{c_0\varphi_{\dot{x}}+c_1Q_1(\varphi)+\varepsilon^2\Big(\gamma_h\varphi_{x\bar{x}\dot{x}}+\gamma h\varphi_{x\bar{x}x}\Big)\\&-\varepsilon^2Q_2(\varphi)\Big\}=\check{y}-\alpha^2\varepsilon^2\check{y}_{x\bar{x}}+\tau\Big\{c_1R_1(\mathrm{w},\bar{\varphi})+c_1Q_1(\mathrm{w})-\varepsilon^2Q_2(\mathrm{w})\Big\}. \qquad (51)\end{aligned}$$

Similarly to (36) the equation (51) yields

$$\begin{aligned}&\|\varphi\|^2+\alpha^2\|\varepsilon\varphi_x\|^2+\tau\gamma h^2\|\varepsilon\varphi_{x\bar{x}}\|^2-\varepsilon^2(c_3/2-c_2)h\sum\varphi_x\varphi_{\bar{x}}\varphi_{\dot{x}}\\&=h\sum\varphi\{\check{y}-\alpha^2\varepsilon^2\check{y}_{x\bar{x}}\}+\tau h\sum\varphi\{c_1R_1(\mathrm{w},\bar{\varphi})+c_1Q_1(\mathrm{w})-\varepsilon^2Q_2(\mathrm{w})\}. \qquad (52)\end{aligned}$$

Again we should assume the existence of a special solution φ, which satisfies the evenness condition (38). The next step is the estimation of the discrepancy w. Assuming the existence of the special even solution y^k of the equation (34) with $k=1,2,...,j-1$, and subtracting one equation (45) (for $s=s_0-1$) from the another one (for $s=s_0$), we obtain:

$$\begin{aligned}&\mathrm{w}-\alpha^2\varepsilon^2\mathrm{w}_{x\bar{x}}+\tau\Big\{c_0\mathrm{w}_{\dot{x}}+c_1R_1(\check{y},\mathrm{w})+\varepsilon^2\Big(\gamma_h\mathrm{w}_{x\bar{x}\dot{x}}+\gamma h\mathrm{w}_{x\bar{x}x}\Big)\\&-\varepsilon^2R_2(\check{y},\mathrm{w})\Big\}=\tau(\check{y}_{\bar{t}}-\alpha^2\varepsilon^2\check{y}_{x\bar{x}\bar{t}})\quad\text{for}\quad s=1, \qquad (53)\\&\mathrm{w}-\alpha^2\varepsilon^2\mathrm{w}_{x\bar{x}}+\tau\Big\{c_0\mathrm{w}_{\dot{x}}+c_1R_1(\bar{\varphi},\mathrm{w})+c_1R_1(\bar{\mathrm{w}},\bar{\bar{\varphi}})+\varepsilon^2\Big(\gamma_h\mathrm{w}_{x\bar{x}\dot{x}}+\gamma h\mathrm{w}_{x\bar{x}x}\Big)\\&-\varepsilon^2R_2(\bar{\varphi},\mathrm{w})\Big\}=-\tau\Big(c_1Q_1(\bar{\mathrm{w}})-\varepsilon^2Q_2(\bar{\mathrm{w}})\Big)\quad\text{for}\quad s>1, \qquad (54)\end{aligned}$$

where $\bar{\mathrm{w}}\overset{\text{def}}{=}\varphi(s-1)-\varphi(s-2)$. Applying the standard techniques we verify the following estimates for φ and w (for the proof see Attachment):

Lemma 4
Let the assumptions of Lemma 3 be satisfied and let φ satisfy the evenness condition (38). Suppose also that

$$\tau \leq q_1 \varepsilon h^2, \quad h \leq q_2 \varepsilon, \tag{55}$$

where constants $q_i > 0$ *are sufficiently small. Then*

$$\|\varphi\|^2_{(2,\tau)}\Big\{1 - (\varepsilon q_1^3 q_2^4)^{1/4}\Big(\|\mathrm{w}\|^2_{(2,\tau)} + (q_1 q_2)^{1/4} h^{3/4}\|\mathrm{w}\|^2_{(1)}\Big)\Big\}$$
$$\leq c_1\|\breve{y}\|^2_{(2)} + c_2\Big\{\|\mathrm{w}\|^4_{(2,\tau)} + \|\mathrm{w}\|^6_{(1)}\Big\}, \tag{56}$$
$$\|\mathrm{w}\|^2_{(2,\tau)} \leq c_3\tau^2\|\breve{y}_{\bar{t}}\|^2_{(1)} \quad for \quad s = 1, \tag{57}$$
$$\|\mathrm{w}\|^2_{(2,\tau)} \leq c_4(\tau\varepsilon^{-2})^2\|\bar{\mathrm{w}}\|^6_{(1)} + c_5\|\bar{\mathrm{w}}\|^4_{(2,\tau)}, \quad for \quad s > 1, \tag{58}$$

where

$$\|f\|^2_{(1)} \overset{\text{def}}{=} \|f\|^2 + \|\varepsilon f_x\|^2, \quad \|f\|^2_{(2)} \overset{\text{def}}{=} \|f\|^2 + \|\varepsilon f_x\|^2 + \|\varepsilon^2 f_{x\bar{x}}\|^2,$$
$$\|f\|^2_{(2,\tau)} \overset{\text{def}}{=} \|f\|^2 + \|\varepsilon f_x\|^2 + \tau h^2\|\varepsilon f_{x\bar{x}}\|^2, \tag{59}$$

$c_i > 0$ denote constants which do not depend on *h, τ, ε, and s.*
In particular, estimates (55)-(58) imply for s = 2

$$\|\mathrm{w}(2)\|^2_{(2,\tau)} \leq c\Big(\tau\varepsilon^{-2}\Big)^2\tau^6 + (c'\tau^2)^2 \leq c\tau^4,$$
$$\|\varphi(2)\|^2_{(2,\tau)}(1 - c\tau^4) \leq c\|\breve{y}\|^2_{(2)} + c'\tau^8. \tag{60}$$

Collecting the estimates (43) and (55)-(58) together, we finally reach the conclusion that the terms of the w-sequence vanish very rapidly,

$$\|\mathrm{w}(s)\|^2_{(2,\tau)} \leq \Big(c'\tau\Big)^{2^s}. \tag{61}$$

Note next that by virtue of (56)-(58), the terms of φ-sequence are bounded uniformly in s

$$\|\varphi(s)\|^2_{(2,\varepsilon)} \leq c\|y^j\|^2 + O(\tau^4). \tag{62}$$

Furthermore, for any $n > 0$

$$\|\varphi(s+n) - \varphi(s)\|_{(2,\varepsilon)} \leq \sum_{i=1}^{n}\|\mathrm{w}_{s+i}\|_{(2,\varepsilon)}$$
$$\leq \|\mathrm{w}_{s+1}\|_{(2,\varepsilon)}\sum_{i=1}^{\infty}\frac{\|\mathrm{w}_{s+i}\|_{(2,\varepsilon)}}{\|\mathrm{w}_{s+1}\|_{(2,\varepsilon)}} \leq c\|\mathrm{w}_{s+1}\|_{(2,\varepsilon)}.$$

This implies the main statement of this subsection:

Theorem 3
Let the assumption Lemma 4 be satisfied. Then the sequence $\varphi(s)$ converges in the $H^2_{\tau,h,\varepsilon}$ sense to the solution of the equation (44). Moreover

$$\|y - \varphi(2)\| \leq c\tau^4, \tag{63}$$

where $H^2_{\tau,h,\varepsilon}$ is the space with the norm (59) and a constant $c > 0$ dos not depend on h, τ, and ε.

3.3 Numerical Simulation
To solve the system of linear equations (45) we use the Gauss method adapted to systems with five non-zero diagonals. This implies the efficiency of the scheme in the sense that it executes *O(I)* arithmetic operations to pass to the next time-level. In accordance with (63) we stop the iterative calculation of *φ(s)* at the second step setting $y^j = \varphi$ (2). Clearly, this implies the appearance of an error. However, our results of numerical simulations justify this decision (see below).

In order to define soliton initial data we solve numerically the equation (14) and define $g(0) = g_*$, where $g_* = g_0$ for $A > 0$ and $g_* = g_1$ for $A < 0$. Next, to avoid the non-uniqueness in the problems (15), (16) we calculate

$$g^*(h) = g_* + \frac{1}{4}h^2\frac{dF}{dg}\Big|_{g=g_*} + \frac{1}{4}\frac{h^4}{4!}\frac{dF}{dg}\frac{d^2F}{dg^2}\Big|_{g=g_*} + \frac{1}{4}\frac{h^6}{6!}\frac{dF}{dg}\left[3\frac{d^3F}{dg^3} + \left(\frac{d^2F}{dg^2}\right)^2\right]\Big|_{g=g_*}$$

and solve the similar (15) (if $A > 0$ or (16) if $A < 0$) problem

$$\frac{dg}{d\eta} = \sqrt{F(g,q)}, \quad \eta \in (h, \infty); \quad g|_{\eta=h} = g^*(h), \tag{64}$$

using the fourth order Runge-Kutta method. The last step is the determination of the profile $\omega(\eta, A)$ in accordance with the rule (8).

Example 1

When $\alpha = c_0 = c_2 = c_3 = 0$, $c_1 = 2$, $\gamma = 1$, and $n = 3$, (1) is the modified KdV equation. For definiteness, here and in what follows we set $\varepsilon = 0.1$. We test the finite difference scheme (45) by calculating the difference Er between the numerical and exact mKdV solitons, and also check the fulfilment of the conservation laws (35) and (36), see Figure 2 and Table 1. The motion of the numerical mKdV soliton is shown in Figure 3.

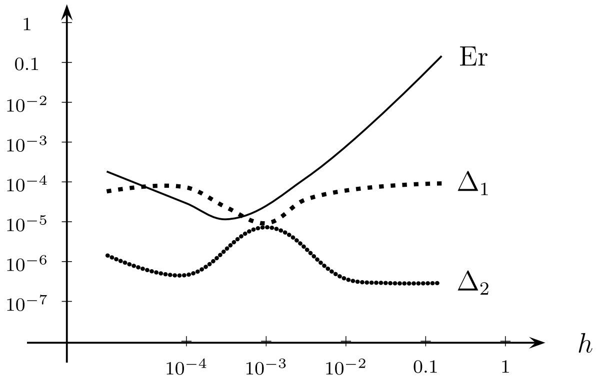

Figure 2: Behavior of the error-functions Er and Δ_k (65) for the mKdV soliton with A = 1.2 and $\tau = h^2$

Here

$$Er = \max_{i=0,\dots,I} |u_{exact}(x_i, t_j) - y_i^j|, \quad \Delta_k = \max_{j'=0,\dots,j} |E_k^{j'} - E_k^0|, \quad k = 1, 2, \tag{65}$$

$u_{\text{exact}} = A\cosh^{-1}(\beta(x - Vt - x^0)/\varepsilon)$ is the exact mKdV soliton, $V = A^2$, $\beta = A$, and y_i^j is the numerical wave at $x_i = x^0 + ih$, $t_j = j\tau$. The energies E_k^j are calculated in accordance with formulas (32), (33),

$$E_1^j = h\sum_{i=0}^{I} y_i^j, \quad E_2^j = \|y^j\|^2 + \alpha^2\|\varepsilon y_x^j\|^2 + \tau^2\sum_{j'=1}^{j}\left\{\|y_{\bar t}^{j'}\|^2 + \alpha^2\|\varepsilon y_{x\bar t}^{j'}\|^2\right\}$$

$$+ \gamma h^2\tau\sum_{j'=1}^{j}\|\varepsilon y_{x\bar x}^j\|^2 + \varepsilon^2(2c_2 - c_3)\tau h\sum_{j'=1}^{j}\sum_{i=0}^{I} y_{ix}^j y_{i\bar x\bar x}^j y_{i\dot x}^j.$$

$h\,(\times 10^{-3})$	20.0	12.5	10.0	7.1	5.5	5.0	4.5	4.1	3.8	3.3
Er $(\times 10^{-3})$	170.7	71.4	46.5	24.0	14.6	1.8	0.9	0.8	7.0	9.0
$\Delta_1\,(\times 10^{-6})$	18.3	14.4	11.1	6.5	3.9	4.1	4.3	2.9	4.6	9.0
$\Delta_2\,(\times 10^{-6})$	0.5	0.4	0.5	0.6	0.6	1.9	1.1	3.5	0.7	1.2

Table 1: The Errors Er, Δ_k for $h \in [3.3 \times 10^{-3}, 0.2]$ and $A = 1.2$ at time $t = t_j = 1$.

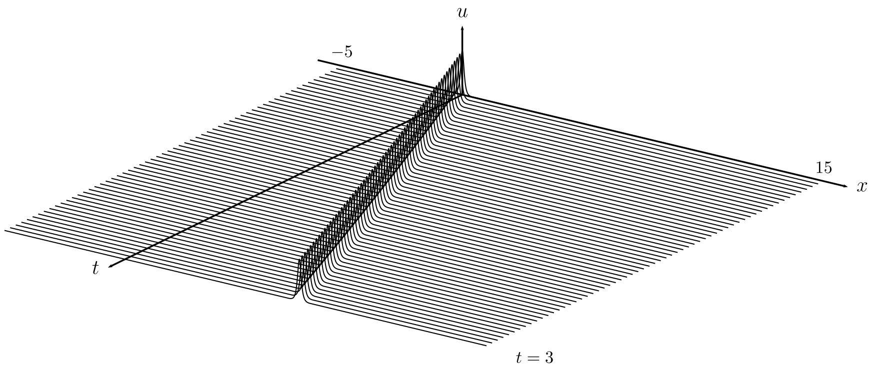

Figure 3: Dynamics of the mKdV Soliton with A = 1.6, h = 4.1×10^{-3}, and τ = h^2

Example 2. Let us consider mgDP equation (1) in the case

$$c_3 = c_2 = 2, \quad \gamma = 2, \quad c_1 = c_0 = \alpha = 1. \tag{66}$$

Then r = 1/2 and Theorem 1 guarantees the existence of solitons with amplitudes under the assumption (18), where $A^*_0 = 0.33$, $A^-_0 = 1.9$, and $A^+_0 = 2.55$.

$h\left(\times 10^{-3}\right)$	20.0	12.5	10.0	7.1	5.5	5.0	4.5	4.1	3.8	3.3
$\Delta_1\left(\times 10^{-5}\right)$	0.5	2.5	7.3	5.2	46.0	20.8	102.3	201.3	523.2	1037.3
$\Delta_2\left(\times 10^{-3}\right)$	328.2	246.1	146.1	97.2	84.1	38.3	1.1	3.5	41.5	317.9

Table 2: The Error Δ_k for $h \in [3.8\times10^{-3}, 0.2]$ and A = 1.2 at time $t = t_j = 1$.

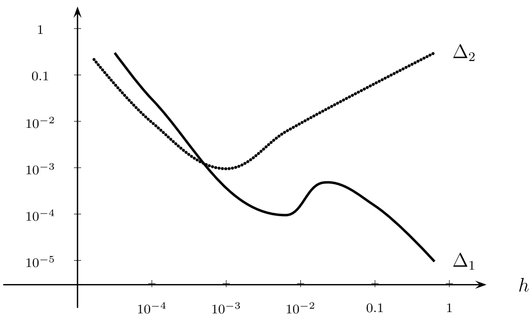

Figure 4: Behavior of the Error- Functions Δ_k for the mgDP Soliton with A = 1.2 in the case (66)

Next, for r = 1/2 the function $F(g,q)$ (10) is the 5-degree polynomial

$$F(g,q) = (1/15)(z-1)^2\Big(-12z^3 + 21z^2 + (20q-6)z + 10q - 3\Big)\Big|_{z=g^{1/2}}.$$

Thus, the real root $g = g^*$ can be found analytically using the Cardano's formula. We set A = 1.2 and define the initial data by solving the problem (64). The graph in Fig. 4 and Table 2 confirm the stability of the wave propagation. Figures 5 and 6 depict the evolution of one and two solitons respectively for the case (66).

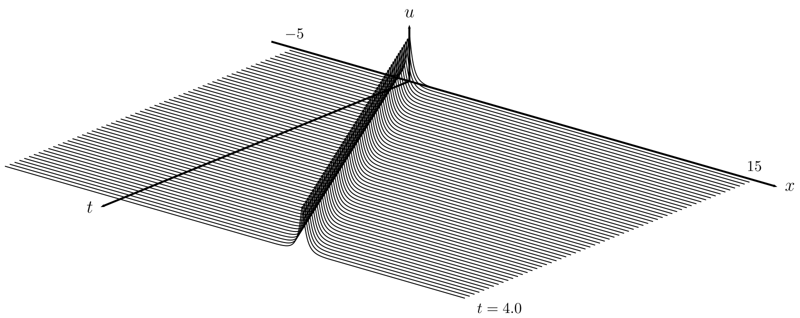

Figure 5: Evolution of the mgDP Soliton with A = 1.2 in the case (66)

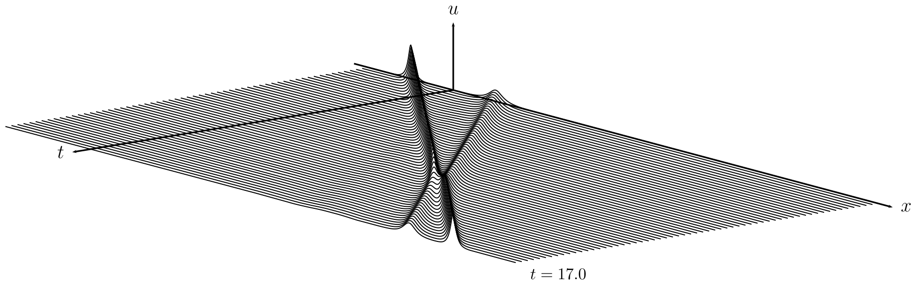

Figure 6: Collision of Two Soliton with $A_1 = 1.2$, $A_2 = 0.5$ in the Case (66)

Example 3: Let us now consider mgDP equation (1) in the case

$$\alpha = c_1 = c_3 = c_2 = 1, \quad \gamma = c_0 = 2. \tag{67}$$

Then $r = 1/2$ and Theorem 1 guarantees the existence of solitons with amplitudes under the assumption (20), where $A* = 0.20$.

$h\left(\times 10^{-3}\right)$	6.2	6.0	5.8	5.7	5.5	5.4	5.2	5.1	5.0
$\Delta_1\left(\times 10^{-4}\right)$	11.4	6.1	16.8	42.1	10.5	18.7	5.0	20.9	51.0
$\Delta_2\left(\times 10^{-3}\right)$	17.0	28.4	23.0	18.6	8.0	7.3	27.3	18.5	10.1

Table 3: The Error Δ_k for h $[5 \times 10^{-3}, 6 \times 10^{-3}]$ and $A = 1.5$ at Time $t = t_j = 1$.

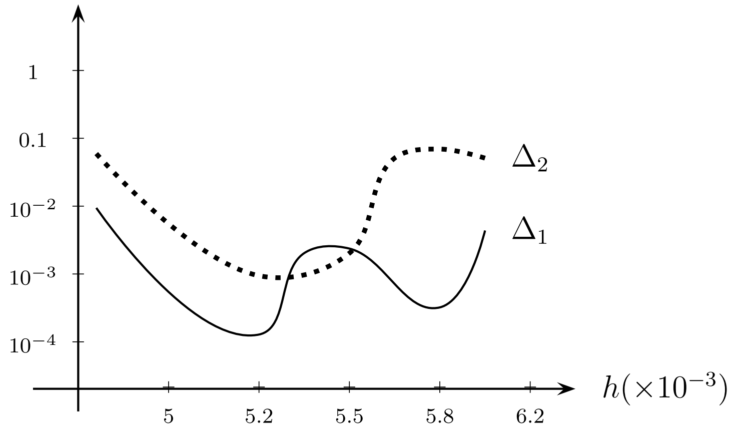

Figure 7: Behavior of the Error-Functions Δk for the mgDP Soliton with A = 1.5 in the Case (67)

Next, similar to Example 2, for $r = 1/2$ the function $F(g, q)$ is the 5- degree polynomial (10). Thus, the real roots $g = g_0$ (0, 1) and $g = g_1 > 1$ can be found analytically using the Cardano's formula. We use either $A > 0$ or $A < 0$ with g_0 or g_1 respectively and define the initial data by solving the problem (64). The graph in Fig. 8 and Table 3 confirm the stability of the antisoliton propagation. Figure 9 depicts soliton-antisoliton interaction for the case (67).

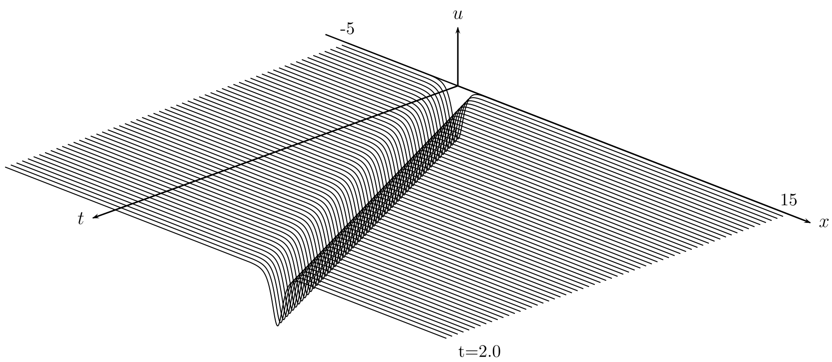

Figure 8: Evolution of the mgDP Antisoliton with $A = -1.8$ in the case (67) with h = 5.2×10^{-3} and $g_1 = 1.73473808$

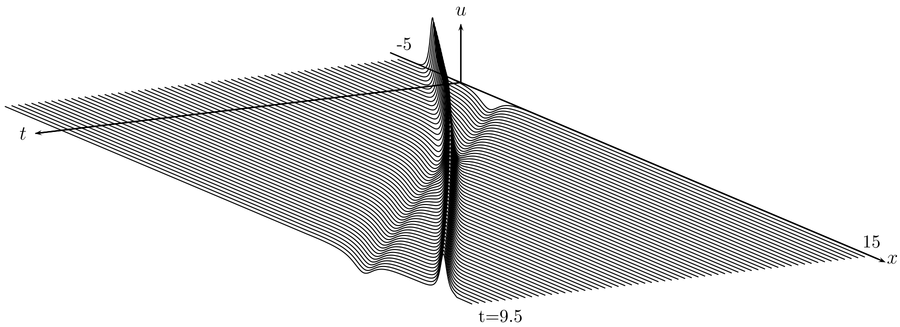

Figure 9: Collision of the soliton-Soliton-Antisoliton with A_1 = 1.8, A_2 = -0.5 in the case (67)

Data Availability

No data was used for the research described in the article.

4. Attachment

In what follows we use the notation

$$||f||_p = \left(h\sum_{i=1}^{I-1}|f_i|^p\right)^{\frac{1}{p}}$$

for the discrete analogs of the $L^p(0, L)$ norm. Again, for simplicity we write

$$||f|| \stackrel{\text{def}}{=} ||f||_p \text{ if } p = 2.$$

Our main tools are the discrete versions of the Ḧolder and the Gagliardo Nirenberg inequalities, namely,

$$h\left|\sum_{i=0}^{N} f_i g_i\right| \leq \|f\|_p\|g\|_q, \quad \frac{1}{p}+\frac{1}{q}=1, \quad 1<p,q<\infty,$$

$$\|\partial_x^r f\|_p \leq c\|f\|^{1-\theta}\Big\{\|f\|^2+\|\partial_x^\ell f\|^2\Big\}^{\theta/2}, \quad \theta\ell = \frac{1}{2}+r-\frac{1}{p}, \quad 0<\theta<1, \ (68)$$

where c is a constant which does not depend on h. Let us recall that the Gagliardo-Nirenberg inequality is the multiplicative form of the embedding theorem [27]. The proof of the discrete version of the Gagliardo- Nirenberg inequality can be found for example in [27]. For $x \in \mathbb{R}^1$ the proof of (68) is trivial. Recall also that for f from the Sobolev space H^l_0 (0, L) of functions with zero value on the boundary,

$$\max_i |f_i| \leq \sqrt{2}\|f\|^{1/2}\|f_x\|^{1/2}. \qquad (69)$$

Furthermore, we use the well-known identities

$$(yg)_x = y_x g + y g_x + h y_x g_x, \quad (yg)_{\bar{x}} = y_{\bar{x}} g + y g_{\bar{x}} - h y_{\bar{x}} g_{\bar{x}},$$

So that

$$(yg)_{\dot{x}} = y_{\dot{x}} g + y g_{\dot{x}} + \frac{h^2}{2}(y_x g_x)_{\bar{x}}. \qquad (70)$$

4.1 Proof of Lemma 3

Multiplying the equation (34) by $\varepsilon^2 y_{xx}$, summing over i and "integrating" by parts we obtain

$$\partial_{\bar{t}}\Big\{\|\varepsilon y_x\|^2+\alpha^2\|\varepsilon^2 y_{x\bar{x}}\|^2\Big\}+\tau\Big\{\|\varepsilon y_{x\bar{x}}\|^2+\alpha^2\|\varepsilon^2 y_{x\bar{x}\bar{t}}\|^2\Big\}$$
$$+\gamma h^2\|\varepsilon^2 y_{x\bar{x}\dot{x}}\|^2 = c_1\varepsilon^2 h\sum Q_1 y_{x\bar{x}} + \varepsilon^4 h\sum Q_2 y_{x\bar{x}}.$$

However, for even y

$$h\sum_{i=1}^{I-1} Q_l(y_i)y_{x\bar{x}} = 0, \quad l = 1,2.$$

Thus, uniformly in j

$$\|\varepsilon y_x^j\|^2 + \alpha^2\|\varepsilon^2 y_{x\bar{x}}^j\|^2 \le c. \tag{71}$$

Next we multiply (34) by $\overline{y}_t$. Summation over i implies the inequality

$$\|y_{\bar{t}}\|^2 + \alpha^2\|\varepsilon y_{x\bar{t}}\|^2 \le c_0\|y_{\bar{t}}\|\|y_x\| + \varepsilon^2\gamma\|y_{x\bar{t}}\|\|y_{x\bar{x}}\| + h|\sum\{c_1Q_1 + \varepsilon^2 Q_2\}y_{\bar{t}}|.$$

Furthermore, applying the Gagliardo - Nirenberg inequalities we conclude

$$\begin{aligned}|h\sum Q_1 y_{\bar{t}}| &\le c\max_i |y|^2\|y_{\bar{t}}\|\|y_x\| \le c\varepsilon^{-2}\|y\|\|\|\varepsilon y_x\|^2\|y_{\bar{t}}\| \\ &\le \frac{1}{4}\|y_{\bar{t}}\|^2 + c\varepsilon^{-4}\Big\{\|y\|^2 + \|\varepsilon y_x\|^2\Big\}^3. \end{aligned}\tag{72}$$

Analogously,

$$\begin{aligned}\varepsilon^2|h\sum Q_2 y_{\bar{t}}| &\le c\varepsilon^2\|y_{x\bar{t}}\|\Big\{\|y_x\|_4^2 + \max_i |y|\|y_{x\bar{x}}\|\Big\} \\ &\le c\varepsilon^{-3/2}\|\varepsilon y_{x\bar{t}}\|\Big\{\|y\|^{3/4}\|\varepsilon^2 y_{x\bar{x}}\|^{5/4} + \|y\|^{1/2}\|\varepsilon y_x\|^{1/2}\|\varepsilon^2 y_{x\bar{x}}\|\Big\} \\ &\le \frac{1}{4}\|\varepsilon y_{x\bar{t}}\|^2 + c\varepsilon^{-3}\Big\{\|y\|^2 + \|\varepsilon y_x\|^2 + \|\varepsilon^2 y_{x\bar{x}}\|^2\Big\}^2. \end{aligned}\tag{73}$$

This and (72) imply the estimate (43)

4.2 Lemma 4 proof

Similarly to (72), (73) we have

$$\begin{aligned}\tau|h\sum Q_1(\mathrm{w})\varphi| &\le c\tau\varepsilon^{-1}\Big\{\|\varphi\|\|\varepsilon \mathrm{w}_x\|\max|\mathrm{w}|^2 + \|\varepsilon\varphi_x\|\|\mathrm{w}\|_6^3\Big\} \\ &\le \frac{1}{8}\Big\{\|\varphi\|^2 + \alpha^2\|\varepsilon\varphi_x\|^2\Big\} + c\tau^2\varepsilon^{-3}\|\mathrm{w}\|_{(2,\tau)}^4, \end{aligned}\tag{74}$$

$$\begin{aligned}\tau\varepsilon^2|h\sum Q_2(\mathrm{w})\varphi| &\le c\tau\|\varepsilon\varphi_x\|\Big(\max|\mathrm{w}_x|\|\varepsilon \mathrm{w}_x\| + \max|\mathrm{w}|\|\varepsilon \mathrm{w}_{xx}\| \\ &\le \frac{1}{8}\|\varphi\|_{(1)}^2 + c\frac{\tau}{\varepsilon h^2}\Big\{1 + \sqrt{\tau}h/\varepsilon\Big\}\|\mathrm{w}\|_{(2,\tau)}^4, \end{aligned}\tag{75}$$

$$\begin{aligned}\tau|h\sum R_1(\mathrm{w},\varphi)\varphi| &\le c\tau\|\varphi\|\Big(\max|\mathrm{w}|^2\|\varphi_x\| + \max|\mathrm{w}|\max|\mathrm{w}_x|\|\varphi\|\Big) \\ &\le c\tau\varepsilon^{-2}\|\mathrm{w}\|_{(1)}^2\|\varphi\|_{(1)}^2 + c\tau^{3/4}(\varepsilon^3 h)^{-1/2}\|\varphi\|^2\|\mathrm{w}\|_{(2,\tau)}^2. \end{aligned}\tag{76}$$

By virtue of the restrictions (55), we pass to the estimate (56).
Next, let us multiply (53) by w. Summing over i and integrating by parts we obtain:

$$\begin{aligned}\|\mathrm{w}\|_{(2,\varepsilon)}^2 \le \tau\Big\{\|\check{y}_{\bar{t}}\|\|\mathrm{w}\| + \alpha^2\|\varepsilon\check{y}_{x\bar{t}}\|\|\varepsilon \mathrm{w}_x\|\Big\} \\ + \tau h\Big|\sum \mathrm{w}\{c_1R_1(\check{y},\mathrm{w}) - \varepsilon^2R_2(\check{y},\mathrm{w})\}\Big|. \end{aligned}\tag{77}$$

Similarly to (74), (75) We have

$$\tau|h\sum \mathrm{w}R_1(\check{y},\mathrm{w})| \le c\tau\varepsilon^{-2}\|\check{y}\|_{(1)}^2\|\mathrm{w}\|_{(1)}^2, \tag{78}$$

$$\tau\varepsilon^2|h\sum \mathrm{w}R_2(\check{y},\mathrm{w})| \le ch^{-1}\sqrt{\tau/\varepsilon}\|\check{y}\|_{(2,\tau)}\|\mathrm{w}\|_{(1)}^2 \le c\sqrt{q_1}\|\check{y}\|_{(2,\tau)}\|\mathrm{w}\|_{(1)}^2. \tag{79}$$

In view of (55), for sufficiently small q_1 we obtain the a-priori estimate (57).
To estimate the discrepancy w for $s > 1$ we should analyze the terms $R_l w$ and $Q_1(\bar{w})w$, l=1, 2. By analogy with (74), (75), and (78) we conclude:

$$\tau|h\sum\Big(c_1Q_1(\bar{\mathrm{w}}) - \varepsilon^2Q_2(\bar{\mathrm{w}}\Big)\mathrm{w}| \le c\tau\max|\bar{\mathrm{w}}|^2\Big\{\|\mathrm{w}_x\|\|\bar{\mathrm{w}}\| + \|\mathrm{w}\|\|\bar{\mathrm{w}}_x\|\Big\}$$
$$+ c\tau\|\mathrm{w}_x\|\Big\{\max|\bar{\mathrm{w}}_x|\|\bar{\mathrm{w}}_x\| + \max|\bar{\mathrm{w}}|\|\bar{\mathrm{w}}_{xx}\|\Big\}$$
$$\le \frac{1}{8}\|\mathrm{w}\|^2_{(2,\tau)} + c\|\bar{\mathrm{w}}\|^4_{(2,\tau)} + c\tau^2\varepsilon^{-4}\|\bar{\mathrm{w}}\|^6_{(1)}, \tag{80}$$
$$c_1\tau|h\sum\Big(R_1(\bar{\varphi},\mathrm{w}) + R_1(\bar{\mathrm{w}},\bar{\bar{\varphi}})\Big)\mathrm{w}| \le c\tau\Big\{\max|\bar{\varphi}|^2\|\mathrm{w}\|\|\mathrm{w}_x\|$$
$$+ \max|\mathrm{w}|^2\|\bar{\varphi}_x\|^2 + \max|\bar{\mathrm{w}}|^2\|\mathrm{w}\|\|\bar{\bar{\varphi}}_x\|^2 + \max|\bar{\mathrm{w}}\bar{\bar{\varphi}}|\Big(\|\mathrm{w}\|\|\bar{\mathrm{w}}_x\| + \|\mathrm{w}_x\|\|\bar{\mathrm{w}}\|\Big)\Big\}$$
$$\le \Big(\frac{1}{8} + c\tau\varepsilon^{-2}(\|\bar{\varphi}\|^2_{(1)} + \|\bar{\bar{\varphi}}\|^2_{(1)})\Big)\|\mathrm{w}\|^2_{(1)} + c(\tau\varepsilon^{-2})^2\|\bar{\mathrm{w}}\|^4_{(1)}. \tag{81}$$

In order to estimate $\mathrm{w}R_2(\bar{\varphi},\mathrm{w})$ we use the same procedure as in (75)

$$\tau\varepsilon^2|h\sum \mathrm{w}R_2(\bar{\varphi},\mathrm{w})| \le c\tau\varepsilon^2\Big\{\max|\bar{\varphi}_x|\|\mathrm{w}_x\|^2 + \max|\varphi|\|\mathrm{w}_x\|\|\mathrm{w}_{xx}\|$$
$$+ \max|\mathrm{w}|\|\bar{\varphi}_{xx}\|\|\mathrm{w}_x\|\Big\} \le c\Big(\sqrt{\tau/\varepsilon}h^{-1} + \tau^{3/4}(\varepsilon\sqrt{h})^{-1}\Big)\|\bar{\varphi}\|_{(2,\tau)}\|\mathrm{w}\|^2_{(2,\tau)}$$
$$\le c\sqrt{q_1}\|\bar{\varphi}\|_{(2,\tau)}\|\mathrm{w}\|^2_{(2,\tau)}. \tag{82}$$

Combining (55) and (80)-(82) yields the desired estimate (58) for the discrepancy w with $s > 1$.

5. Conclusion

Energy estimates and results of numerical experiments confirm the adaptation of the balance laws (2), (3) for the gmKdV equation (1) by difference scheme (45). This implies the stability of the motion of even-shaped waves in both the analytical and numerical sense. Moreover, the scheme remains stable even in the case of soliton interaction, when, generally speaking, the solution ceases to be even at the time instant of collision of waves [38]. At the same time, it turned out that equation (1) with n = 3 is much more sensitive to the accuracy of the initial data than the gDP equation with n = 2. Indeed, the gmKdV equation requires accuracy $O(h8)$ in the initial data approximation for (64), while for the gDP equation it was sufficient to use only three terms of the Taylor expansion, see [23].

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