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Cauchy – Riemann Conditions for the Maxwell's Equations of a Single-Frequency Quaternion

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Abstract

As is known, hypercomplex numbers have both a scalar part and imaginary parts. Unlike the well-known Maxwell equations, the equations written for a quaternion in 4D hypercomplex space also have a scalar part. Since these equations are obtained mathematically by multiplying a quaternion in vector representation by a differential operator of a quaternion in matrix representation, the quaternion is a solution to this equation.

It is shown that by solving the presented equations, it is possible to obtain two types of waves: magnetic and electric, using, respectively, magnetic and electric intensities. Quaternion waves contain particles that are formed from magnetic or electric intensities through the operation of scalar multiplication by the Hamiltonian operator. Magnetic waves have charged electrons as particles, and electric waves have electron spins (rotors).

The obtained equations satisfy the Cauchy-Riemann conditions and, consequently, the requirements of conservation of energy during transformations.

Keywords: Maxwell's Equations, Quaternion, Circulation, Rotor, Cauchy-Riemann Conditions

1. Introduction

In 1873, Maxwell united electric and magnetic fields into a single theory of electromagnetism based on four fundamental equations. An important conclusion of this theory was the prediction of electromagnetic waves that propagate at the speed of light. These predictions were confirmed by Hertz's experiments in 1887. The discovery of electromagnetic waves and the development of electronic devices to generate them revolutionized telecommunications in the 20th century.

However, subsequent physical experiments showed that space and time form a single system that does not satisfy the Galilean transformation for inertial systems. The unity of space and time is substantiated in the theory of relativity on the basis of the Lorentz transformation.

Maxwell's equations for electromagnetic waves, obtained in 3D space, clearly did not show the relativity of space and time. However, Maxwell's equations, obtained for a quaternion in 4D space, use 4 coordinate axes, one of which is scalar and the other 3 are imaginary. Imaginary axes in 3D space have the dimension of frequencies and, accordingly, form a connection between frequency, i.e. the time scale, and the location of a point in space [1]. Moreover, this idea is consistent with M. Planck's hypothesis about the dependence of the energy of elementary particles on frequency.

Physical experiments have also shown the discreteness of the atomic radiation spectrum, which does not follow from Maxwell's equations. As a result, quantum theory emerged, which proves that the energy of elementary particles changes in jumps or quanta. When deriving Maxwell's equations using a quaternion in 4D space, the electron is represented as a 4D vector with one scalar part and three imaginary parts. It has been shown that when the law of conservation of energy is fulfilled, the spatial movement of electrons occurs in jumps and their movement occurs in an orbit without loss of energy and the action of gravitational forces [1].

The aim of the article is to show that Maxwell's equations for a single-frequency quaternion satisfy the Cauchy-Riemann conditions and, consequently, the law of conservation of energy.

2. Materials and Methods for Solving the Problem

Let us write the quaternion in algebraic representation form as

$$q = s + ix + jy + kz\,, \tag{1}$$

where s, x, y z – real numbers, i, j, k – imaginary units.

Let real numbers change their values with the change of time $x = y = z = \omega_c t$ where ω_c – angular frequency of the carrier.

When transforming the quaternion (1) using any function f, we also obtain a quaternion in the form of a sum of functions with a scalar and three imaginary parts:

$$f(t) = p(t) + iu(t) + jv(t) + kw(t)\,. \tag{2}$$

As functions $p(t)$, $u(t)$, $v(t)$ and $w(t)$ in (2) one can consider any continuous functions with a bounded norm, for example, pulses of finite length.

According to Euler's formula, the quaternion (1) with unit modulus $|q| = 1$ can be represented in trigonometric form for the angular frequency ω_c as

$$q(t) = e^{\hat{i}\omega_c t} = \cos\omega_c t + \hat{i}\sin\omega_c t\,, \tag{3}$$

where $\hat{i} = (i + j + k)/\sqrt{3}$ – imaginary unit of a single-frequency quaternion.
To get rid of imaginary units, we represent the quaternion (1) in algebraic form as a 4×4 matrix [2]:

$$\mathbf{Q} = \begin{bmatrix} s & x & y & z \\ -x & s & -z & y \\ -y & z & s & -x \\ -z & -y & x & s \end{bmatrix}. \tag{4}$$

We decompose matrix (4) into *basis matrices* **E, I, J, K** and write quaternion (1) in algebraic form as the sum of matrices:

$$\mathbf{Q} = \mathbf{E}s + \mathbf{I}x + \mathbf{J}y + \mathbf{K}z\,, \tag{5}$$

where

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ \mathbf{I} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{6}$$

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},\ \mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding basis matrices (6) of the quaternion, as well as the elements i, j and k, are related by the multiplication rules presented in Table 1.

Table 1. Multiplication of basis matrices.

×	**E**	**I**	**J**	**K**
E	**E**	**I**	**J**	**K**
I	**I**	–**E**	**K**	–**J**
J	**J**	–**K**	–**E**	**I**
K	**K**	**J**	–**I**	–**E**

We write the imaginary part $\hat{\imath}$ of the quaternion (5) in matrix representation as an *imaginary matrix*:

$$\hat{\mathbf{I}} = (\mathbf{I} + \mathbf{J} + \mathbf{K}) / \sqrt{3} . \tag{7}$$

According to Table 1, the basis matrices are orthogonal: $\mathbf{II}^{\mathrm{T}} = \mathbf{E}$, $\mathbf{JJ}^{\mathrm{T}} = \mathbf{E}$, $\mathbf{KK}^{\mathrm{T}} = \mathbf{E}$. Moreover, the basis matrices of a quaternion are also quaternions and do not intersect in space when superimposed. Therefore, each basis matrix forms a 4D spatial coordinate axis in 4D space.
The formation of a wave can be represented by a dynamic equation in the state space [2]:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) , \tag{8}$$

where $\mathbf{A} = \omega_c \hat{\mathbf{I}}$ – *state transition matrix*.

Note that representation (8) is also consistent with M. Planck's hypothesis on the dependence of the energy of waves emitted or absorbed by particles during quantum jumps on frequency, in our case ω_c.

The solution to the matrix differential equation (8) is the exponential (3) in matrix representation, which is called the *fundamental matrix*:

$$e^{\hat{\mathbf{I}}\omega_c t} = \mathbf{\Phi}(\omega_c, t) = \mathbf{E}\cos(\omega_c t) + \hat{\mathbf{I}}\sin(\omega_c t) . \tag{9}$$

The fundamental matrix (9) is orthogonal, since $\mathbf{\Phi}(\omega_c,t)\mathbf{\Phi}^{\mathrm{T}}(\omega_c,t) = \mathbf{\Phi}^{\mathrm{T}}(\omega_c,t)\mathbf{\Phi}(\omega_c,t) = \mathbf{E}$. An orthogonal matrix does not change the vector modulus when multiplied, the rank of the fundamental matrix is 4, and the determinant of the fundamental matrix is

$$|\mathbf{\Phi}(\omega_c,t)| = \det[\mathbf{\Phi}(\omega_c,t)] = (\cos^2 \omega_c t + \sin^2 \omega_c t)^2 = 1 .$$

The product of quaternions q and p can be represented as [3]:

$$qp = [q_0, \mathrm{q}][p_0, \mathrm{p}] = (q_0 p_0 - \mathrm{q} \,.\, \mathrm{p}) + (q_0\,\mathrm{p} + p_0\,\mathrm{q} + \mathrm{q} \times \mathrm{p}) , \tag{10}$$

where q and p are the corresponding pure quaternions, q.p is the scalar product and q × p is the vector product of pure quaternions.

Using expression (10) and representing the product of quaternions as the product of the derivative of the conjugate quaternion with respect to time in matrix form by the vector of the quaternion function, the equation is obtained [1]:

$$\mathbf{Q}^{\mathrm{T}}\mathbf{p} = \begin{bmatrix} \partial_{s,t}p - \partial_{x,t}u - \partial_{y,t}v - \partial_{z,t}w \\ \partial_{x,t}p + \partial_{s,t}u + \partial_{z,t}v - \partial_{y,t}w \\ \partial_{y,t}p - \partial_{z,t}u + \partial_{s,t}v + \partial_{x,t}w \\ \partial_{z,t}p + \partial_{y,t}u - \partial_{x,t}v + \partial_{s,t}w \end{bmatrix} \Leftrightarrow \begin{bmatrix} \text{scalar} \\ \partial_{s,t}p - \nabla.\mathrm{f}\,(q) \\ \text{circulation} \\ \partial_{x,t}p + \partial_{s,t}u - (\partial_{y,t}w - \partial_{z,t}v) \\ \partial_{y,t}p + \partial_{s,t}v - (\partial_{z,t}u - \partial_{x,t}w) \\ \underbrace{\partial_{s,t}p}_{\nabla p} + \underbrace{\partial_{s,t}w}_{\partial_s \mathrm{f}(q)} - \underbrace{(\partial_{x,t}v - \partial_{y,t}u)}_{\nabla\times\mathrm{f}(q)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \tag{11}$$

where $\nabla = [\partial_x \quad \partial_y \quad \partial_z]^{\mathrm{T}}$ – Hamilton operator in vector representation, $\mathrm{f}(q) = [u \quad v \quad w]^{\mathrm{T}}$ – vector of pure quaternions of the function of quaternion (2).

Note that each component of circulation shows a difference in the rates of change in different directions, which corresponds to the tendency of rotation in one of the coordinate planes. If there is a circulation of the field at the point of the *x*-component, then it means that the field has a circulation around this point in the YZ plane.

Since the derivative in (11) is taken over the conjugate quaternion, then based on the Cauchy-Riemann conditions (CRC), the elements of the resulting vector (11) must be equal to zero. Obviously, the solution to this vector equation will be a quaternion. Consequently, in the obtained vector (11), instead of a function of a quaternion $f(q)$ and, accordingly, a pure quaternion f(q), one can use the vectors of electric **E** or magnetic intensity **H** as quaternions or as pure quaternions E and H.

Since the electric flux $\mathrm{D} = \varepsilon_0 \mathrm{E}$, where ε_0 is the permittivity of free space (electric constant), and the magnetic flux B is related to the magnetic intensity Has $\mathrm{B} = \mu_0 \mathrm{H}$, where μ_0 is the magnetic permeability of free space and $\sqrt{\mu_0 \varepsilon_0} = c$ is the speed of light, then in expression

(11) one can also use the electric D or magnetic B fluxes with the corresponding permeabilities.

Thus, using the operation of multiplying quaternions in vector representation and using the CRC, a quaternion vector is obtained, the elements of which consist of a scalar part and a vector part. The CRC correspond to the law of conservation of energy, therefore, the elements of the resulting vector must be equal to zero. Using this expression, Maxwell's equations for magnetic and electric waves were obtained [1]. Since the equality to zero of vector (11) was obtained using a quaternion, the solution to the vector equation will also be a quaternion.

3. Calculation of Cauchy-Riemann Conditions for a Single-Frequency Quaternion

Using the expressions presented above for a single-frequency quaternion, we obtain the CRC for magnetic and electric waves.

3.1 Cauchy-Riemann Conditions for Magnetic Waves of a Single-Frequency Quaternion

For magnetic intensity H, expression (11) is represented as [1]:

$$\begin{bmatrix} \overset{\text{Gauss's law for H}}{\underbrace{\partial_{s,t} p_{\mathbf{H}}}_{\rho_m} - \nabla.\mathrm{H}} \\ \overbrace{\partial_{x,t} p_{\mathbf{H}}}^{\text{scalar } \mathbf{H}} + \overbrace{\partial_{s,t} u_{\mathrm{H}}}^{\text{Lenz's rule}} - \overset{\text{circulation vector H}}{(\partial_{y,t} w_{\mathrm{H}} - \partial_{z,t} v_{\mathrm{H}})} \\ \partial_{y,t} p_{\mathbf{H}} + \partial_{s,t} v_{\mathrm{H}} - (\partial_{z,t} u_{\mathrm{H}} - \partial_{x,t} w_{\mathrm{H}}) \\ \underbrace{\partial_{z,t} p_{\mathbf{H}}}_{\nabla p_{\mathbf{H}}} + \underbrace{\partial_{s,t} w_{\mathrm{H}}}_{\partial_s \mathrm{H}} - \underbrace{(\partial_{x,t} v_{\mathrm{H}} - \partial_{y,t} u_{\mathrm{H}})}_{\nabla \times \mathrm{H}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{12}$$

Using expression (9), we write the magnetic intensity as the product of the initial state vector **h** and the fundamental matrix to the power *n*:

$$\mathbf{H}(n,\omega_c,t,\mathbf{h}) = \mathbf{\Phi}(n,\omega_c,t)\mathbf{h} = \left[\mathbf{E}\cos(n\omega_c t) + \hat{\mathbf{I}}\sin(n\omega_c t)\right]\mathbf{h}\ , \tag{13}$$

where $\mathbf{h} = \begin{bmatrix} h_s & h_x & h_y & h_z \end{bmatrix}^{\mathrm{T}}$ – vector of initial state, $\boldsymbol{h_s}$ – scalar part, $\mathrm{h} = \begin{bmatrix} h_x & h_y & h_z \end{bmatrix}^{\mathrm{T}}$ – imaginary part as a pure quaternion.

The tension vector (13) is a quaternion, where the imaginary part is a pure quaternion H. For a single-frequency quaternion, the frequency on the three imaginary axes of 3D space will be the same. Therefore, the frequency space is homogeneous and the elements of the initial state vector must also be the same.

Let the elements of **h** be equal to *h*. Then, expression (13) can be simplified and written as:

$$\mathbf{H}(n,\omega_c,t,h) = h\begin{bmatrix} \cos(n\omega_c t) + \sqrt{3}\sin(n\omega_c t) \\ \cos(n\omega_c t) - \sin(n\omega_c t)/\sqrt{3} \\ \cos(n\omega_c t) - \sin(n\omega_c t)/\sqrt{3} \\ \cos(n\omega_c t) - \sin(n\omega_c t)/\sqrt{3} \end{bmatrix}. \tag{14}$$

Let us consider the vector $\mathbf{h} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$ as a vector of initial values of magnetic intensity. Figure 1 shows the elements of the vector (14), the scalar part is shown in red, and the same elements of the vector part are shown in blue and dotted.

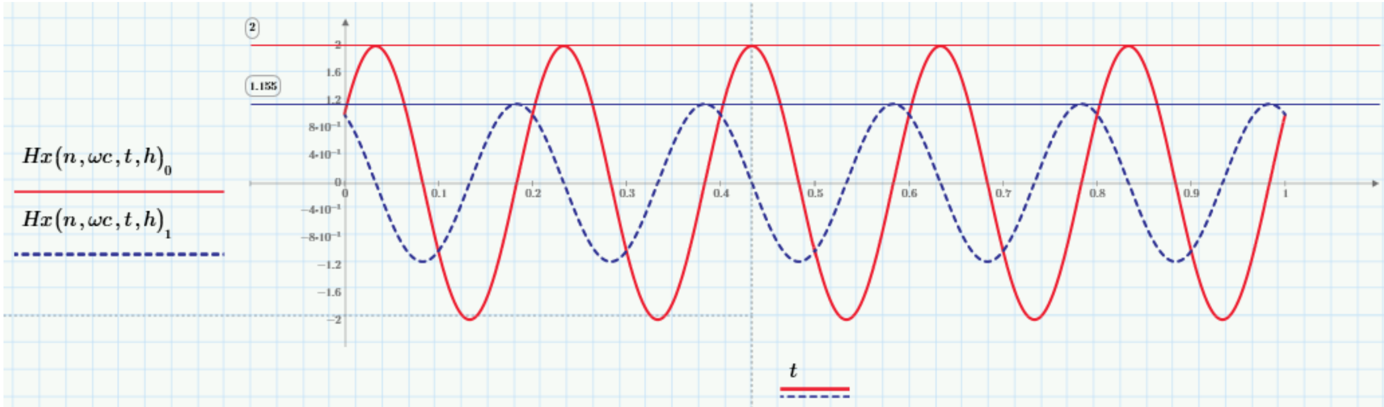

Figure 1: Elements of the Magnetic Intensity Vector of a Single-Frequency Quaternion

For ease of consideration, we shift the initial phases of the obtained oscillations in such a way as to obtain the maximum value of the scalar part at $t = 0$. The initial phase of the scalar part is calculated as $\varphi = \text{atan}(\sqrt{3}) = 60°$. Accordingly, the amplitude of the scalar part when shifted by an angle φ will be equal to the value at $t = 0$, $A_0 = h\left(\cos(\varphi) + \sqrt{3}\sin(\varphi)\right)$. In our case $A_0 = \pm 2$.

Since all imaginary components of the magnetic intensity will be shifted by π/2 relative to the scalar part, the initial phase of the imaginary parts will be $\theta = \pi/2 + \text{atan}(-1/\sqrt{3}) = \pi/2 - 30° = 60°$, and the amplitude is determined at $t = 0$ as $A_1 = \cos(\theta - \pi/2) - \frac{\sin(\theta - \pi/2)}{\sqrt{3}}$, i.e. without rotation by $\pi/2$. In our case, $A_1 = \pm 1.155$. In Figure 1, the amplitude values are shown by horizontal markers.

Figure 2 shows the graphs of the scalar part (red) and the imaginary part (blue, dotted line) with a phase shift of $\varphi = \theta$, respectively. In this case, the scalar part is divided proportionally between the three coordinate axes and will be equal in amplitude to the imaginary parts on the axes.

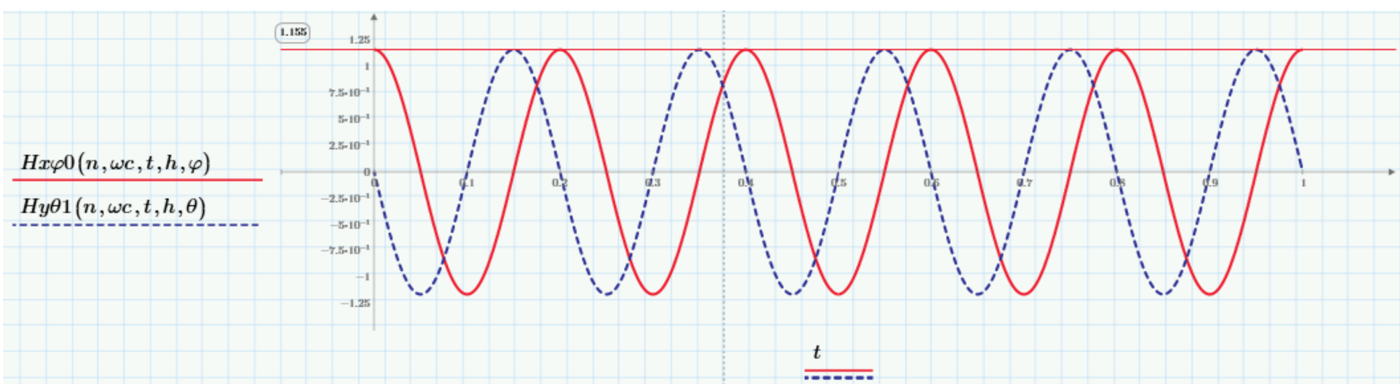

Figure 2: Elements of the Magnetic Intensity Vector of a Single-Frequency Quaternion with a Phase Shift

Based on (11) for the magnetic intensity vector H and, accordingly, the pure quaternion H, the following two equations were obtained [1]:

$$\nabla.\mathrm{H} = \rho_m \text{ – scalar equation,} \tag{15}$$

$$\nabla \times \mathrm{H} = \partial_{s,t}\mathrm{H} + \nabla p_{\mathrm{H}} \text{ – vector equation.} \tag{16}$$

Note that in Maxwell's equations in 3D space $\nabla.\mathrm{H} = 0$, however in 4D quaternion space $\nabla.\mathrm{H} \neq 0$, since there is a scalar coordinate axis. The scalar product (15) of the Hamiltonian operator on the vector of the imaginary part of the magnetic intensity, i.e. the pure quaternion H, is analogous to Gauss's law for charges and shows the equality of the mass density of rotors (electron spins) ρ_m in 4D space.

Let us calculate the values of equation (15) for the magnetic intensity function (14). It is necessary to take into account that the derivatives are taken along the orthogonal axes x, y, z and are added as vectors. We write the scalar product $\nabla.\mathrm{H}$ as follows:

$$\nabla.\mathrm{H} = \nabla^{\mathrm{T}}\, \mathrm{H}(n, \omega_c, t, h) = \begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} h \begin{bmatrix} \cos(n\omega_c t) - \sin(n\omega_c t)/\sqrt{3} \\ \cos(n\omega_c t) - \sin(n\omega_c t)/\sqrt{3} \\ \cos(n\omega_c t) - \sin(n\omega_c t)/\sqrt{3} \end{bmatrix} =$$

$$= h\frac{\partial}{\partial t}\left(\sqrt{3}\cos(n\omega_c t) - \sin(n\omega_c t)\right) = -hn\omega_c\left(\cos(n\omega_c t) + \sqrt{3}\sin(n\omega_c t)\right). \qquad (17)$$

Let us calculate the value of the mass density of the rotors using the formula in (12) $\partial_{s,t} p_{\mathbf{H}}$. As can be seen, the time derivative is taken along the scalar axis s. Therefore, it is necessary to rotate the intensity vector (14) using the imaginary matrix unit (7), isolate the scalar part and take the derivative with respect to time. Since the axes are orthogonal, the point mass density of the rotors is divided equally between the imaginary axes. As a result, we get (17):

$$\rho_m = \partial_{s,t} p_{\mathbf{H}} = h\frac{\partial}{\partial t}\left(\sqrt{3}\cos(n\omega_c t) - \sin(n\omega_c t)\right) = -hn\omega_c\left(\cos(n\omega_c t) + \sqrt{3}\sin(n\omega_c t)\right).$$

Thus, we have obtained that the quaternion function (14) satisfies equation (15) for any value of the quaternion degree n.

Let us calculate the power of the waves of a single-frequency quaternion (14) at the initial state $\mathbf{h} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$. The initial state vector has power 4. Power of a scalar $P_0 = \frac{1}{T}\int_0^T \left(\cos(n\omega_c t) + \sqrt{3}\sin(n\omega_c t)\right)^2 dt = 2$. The power of each vector function is $P_1 = \frac{1}{T}\int_0^T \left(\cos(n\omega_c t) - \frac{1}{\sqrt{3}}\sin(n\omega_c t)\right)^2 dt = \frac{2}{3}$, which adds up to 2. Consequently, the total power of all elements of the intensity vector will be equal to the power of the initial state vector **h**. This result confirms the orthogonality of the fundamental matrix $\mathbf{\Phi}(n, \omega_c, t)$. In addition, the equality to zero of the difference between the power of the scalar part and the sum of the powers of the vector part indicates the fulfillment of the CRC. It is also seen that the power of the scalar part is distributed equally across the three imaginary axes.

The vector part (16) corresponds to Maxwell's equation in 3D for the circulation of magnetic intensity. However, instead of the induced magnetic intensity (eddy) current J in the conductor, the electromotive force (EMF) $\partial_{s,t}$ that creates this current is shown. In addition, as is known, Maxwell added electric current to Ampere's equation, obtained by changing the electric flux D over time. In (16) it is mathematically shown that electric current is formed by changing the scalar part of the magnetic intensity vector p_{H} along all three coordinate axes x, y, z [1].

Let us calculate the execution of the CRC for the vector equation (16) in the x direction of the imaginary coordinate axis. Let us represent equation (11) for magnetic intensity in the x direction as:

$$(\partial_{x,t} p_{\mathbf{H}} + \partial_{s,t} u_{\mathrm{H}}) + (\partial_{z,t} v_{\mathrm{H}} - \partial_{y,t} w_{\mathrm{H}}) = 0. \qquad (18)$$

The circulation along the x-axis, i.e. in the YZ plane, is equal to $(\partial_{z,t} v_{\mathrm{H}} - \partial_{y,t} w_{\mathrm{H}})$.

The partial time derivative of the field strength vector v_{H} in the z-axis direction on the YZ plane is calculated as

$$\partial_{z,t} v_{\mathrm{H}} = h\partial_{z,t}\left(\cos(n\omega_c t) - \frac{1}{\sqrt{3}}\sin(n\omega_c t)\right) = -hn\omega_c\left(\frac{1}{\sqrt{3}}\cos(n\omega_c t) + \sin(n\omega_c t)\right). \qquad (19)$$

The y-axis is perpendicular to the z-axis, so to calculate the circulation for the same direction, we first shift it by $-\pi/2$ and then calculate the time derivative of the magnetic intensity:

$$\partial_{y,t} w_{\mathrm{H}} = h\partial_{y,t}\left(\cos(n\omega_c t - \pi/2) - \frac{1}{\sqrt{3}}\sin(n\omega_c t - \pi/2)\right) = hn\omega_c\left(\cos(n\omega_c t) - \frac{1}{\sqrt{3}}\sin(n\omega_c t)\right). \qquad (20)$$

Figure 3 shows the circulation $(\partial_{z,t} v_{\mathrm{H}} - \partial_{y,t} w_{\mathrm{H}})$ as a gradient. The direction of the current created by the circulation of magnetic field in the YZ plane in the x direction is determined by the gimlet or right-hand rule and is shown in the figure as an arrowhead.

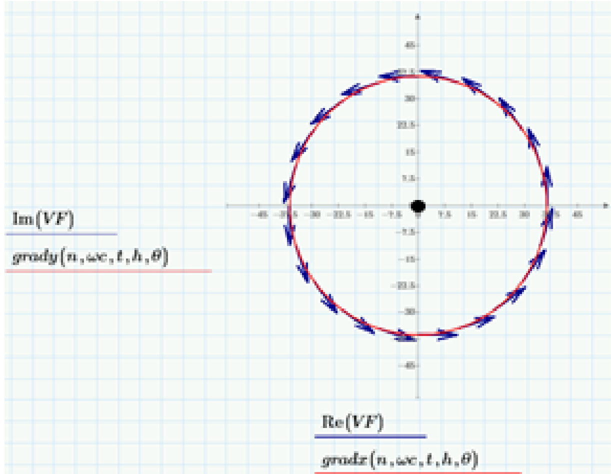

Figure 3: Circulation of Magnetic Intensity v_H and w_H in the YZ Plane Along the x Axis

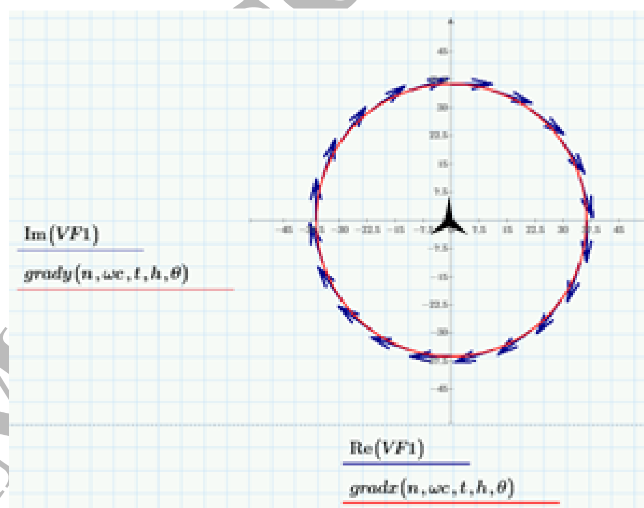

Figure 4: Circulation of Magnetic Intensity p_H and u_H Along the x-Axis

Let us consider the second part of equation (18) $(\partial_{x,t}p_{\mathbf{H}} + \partial_{s,t}u_{\mathrm{H}})$ in the x direction. The first partial derivative with respect to time is taken from the scalar part of the quaternion (14) $p_{\mathbf{H}}$ in the x direction, and the second partial derivative is taken from the vector part u_{H}, which is also directed along the x axis, and is mapped onto the scalar axis s.

Let's find the time derivative along the x-axis of the scalar part of the quaternion $p_{\mathbf{H}}$ divided by power by 3:

$$\partial_{x,t}p_{\mathbf{H}} = h\partial_{x,t}\left(\frac{\cos(n\omega_c t) + \sqrt{3}\sin(n\omega_c t)}{\sqrt{3}}\right) = hn\omega_c\left(\cos(n\omega_c t) - \frac{1}{\sqrt{3}}\sin(n\omega_c t)\right). \qquad (21)$$

Let's calculate the partial derivative of magnetic intensity with respect to time :

$$\partial_{s,t}u_{\mathrm{H}} = h\partial_{s,t}\left(\cos(n\omega_c t) - \frac{1}{\sqrt{3}}\sin(n\omega_c t)\right) = -hn\omega_c\left(\frac{1}{\sqrt{3}}\cos(n\omega_c t) + \sin(n\omega_c t)\right) \qquad (22)$$

Comparing (20) and (21), as well as (19) and (22), we see that $\partial_{y,t}w_{\mathrm{H}} = \partial_{x,t}p_{\mathbf{H}}$ and $\partial_{z,t}v_{\mathrm{H}} = \partial_{s,t}u_{\mathrm{H}}$. Therefore, according to equation (18), we can depict $(\partial_{x,t}p_{\mathbf{H}} + \partial_{s,t}u_{\mathrm{H}})$ in the polar coordinate system as a circulation with the opposite direction of the gradient, as shown in Figure 4. With such rotation we obtain a current with the opposite direction to the current shown in Figure 3. The direction of current in Figure 4 is shown as the fletching of an arrow. Since the magnitudes of the currents coincide and the directions are opposite, they are compensated and, therefore, the CRC is performed.

In terms of the rotor operator, the circulation for any differential loop can be written compactly as $C = (\nabla \times \mathbf{H}) \cdot \mathbf{dS}$ where $\mathbf{dS} = \mathbf{n}dS$ is the area of the element in the direction of the normal vector **n**, perpendicular to the area of the contour in accordance with the gimlet or right-hand rule.

The radius of the circumference of the described circle, shown in Figures 3 and 4, is calculated as $R = A_1\omega_c hn$, where A_1 is the amplitude of the elements of the magnetic intensity vector (14), taking into account the division of the scalar part by power equally into three imaginary axes. The area of the described circle is calculated as $S = \pi R^2 = \pi\left(A_1\omega_c hn\right)^2$.

From the expressions of the vector part (11) it is evident that the directions of the partial derivatives along the imaginary axes $\nabla p_{\mathbf{H}}$ are equal to the corresponding directions of the currents formed by the circulations $\nabla \times \mathrm{f}(q)$ along these axes. At the same time, the scalar axis s is orthogonal to all imaginary axes x, y, z. Therefore, the expressions for circulation along all axes will be the same.

Thus, we have obtained the execution of the CRC for the x-axis direction. Similar transformations for the y and z directions in (11) gave the same result. From this we can conclude that the CRC are fulfilled for the function of a single-frequency quaternion of magnetic intensity (14) for any power n.

3.1 Cauchy-Riemann Conditions for Electric Waves of a Single-Frequency Quaternion

For electrical intensity E, expression (11) is represented as [1]:

$$\begin{bmatrix} \overset{\text{Gauss's law for E}}{\underbrace{\partial_{s,t} p_{\mathbf{E}}}_{\rho_q} - \nabla.\mathrm{E}} \\ \overbrace{\partial_{x,t} p_{\mathbf{E}}}^{\text{scalar }\mathbf{E}} + \overbrace{\partial_{s,t} u_{\mathrm{E}}}^{\text{EMF of self-induction}} - \overset{\text{circulation vector E}}{(\partial_y w_{\mathrm{E}} - \partial_z v_{\mathrm{E}})} \\ \partial_{y,t} p_{\mathbf{E}} + \partial_{s,t} v_{\mathrm{E}} - (\partial_z u_{\mathrm{E}} - \partial_x w_{\mathrm{E}}) \\ \underbrace{\partial_{z,t} p_{\mathbf{E}}}_{\nabla p_{\mathrm{E}}} + \underbrace{\partial_{s.t} w_{\mathrm{E}}}_{\partial_{s,t}\mathrm{E}} - \underbrace{(\partial_x v_{\mathrm{E}} - \partial_y u_{\mathrm{E}})}_{\nabla\times\mathrm{E}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{23}$$

where $p_{\mathbf{E}}$, u_{E}, v_{E}, w_{E} are functions of electrical intensity in the form of a quaternion (2), $\mathrm{E} = \begin{bmatrix} u_{\mathrm{E}} & v_{\mathrm{E}} & w_{\mathrm{E}} \end{bmatrix}^{\mathrm{T}}$ is a pure quaternion of the electrical intensity vector **E**.

Based on the CRC (11), which sets the requirements of the law of conservation of energy, each element of the vector (23) must be equal to 0. Therefore, from (23) we obtain the following two equations:

$$\nabla.\mathrm{E} = \rho_q \text{ – scalar equation,} \tag{24}$$

$$\nabla\times\mathrm{E} = \partial_{s,t}\mathrm{E} + \nabla p_{\mathbf{E}}. \text{ – vector equation.} \tag{25}$$

Expression (24) represents Gauss's law, which relates electrical intensity to charge density. In the resulting representation (25), instead of the time-varying magnetic flux $\partial_t \mathbf{B}$, the EMF $\partial_{s,t}\mathrm{E}$ is written, which creates a current in a closed circuit according to Faraday's law. In expression (25), in contrast to Maxwell's equation, there is additionally a vector ∇p_{E} of derivatives of the scalar part of the electrical intensity along the imaginary coordinate axes in the matrix representation. This is explained by the fact that Maxwell's equations are written for 3D space and do not take into account the scalar part.

According to (13) and (14), the electric field vector is a quaternion:

$$\mathbf{E}(n,\omega_c,t,h) = h\begin{bmatrix} \cos(n\omega_c t) + \sqrt{3}\sin(n\omega_c t) \\ \cos(n\omega_c t) - \sin(n\omega_c t)/\sqrt{3} \\ \cos(n\omega_c t) - \sin(n\omega_c t)/\sqrt{3} \\ \cos(n\omega_c t) - \sin(n\omega_c t)/\sqrt{3} \end{bmatrix}. \tag{26}$$

The graphs of the elements of vector (26) will be similar to the graphs shown in Figures 1 and 2 for magnetic intensity. The scalar equation (24) for the electric field strength is similar in form to equation (15) for the magnetic field strength and shows that the mass density of the electron charges ρ_q is equal to the scalar product of the electric field strength and the Hamiltonian operator in 4D space.

Since the quaternion of electric intensity (26) is the same as the quaternion of magnetic intensity (14), then according to (17) the quaternion function (26) also satisfies equation (24) and, consequently, the law of conservation of energy, i.e. the law of conservation of energy for any value of the degree of the quaternion *n*.

Similarly, according to equation (18), it can be shown that equation (25) for the electric field strength in the *x*-axis direction is:

$$(\partial_{x,t} p_{\mathbf{E}} + \partial_{s,t} u_{\mathrm{E}}) + (\partial_{z,t} v_{\mathrm{E}} - \partial_{y,t} w_{\mathrm{E}}) = 0\,.$$

The circulation of electrical intensity will have the form shown in Figures 3 and 4 for magnetic intensity. Having calculated the circulations for the *y* and *z* axis directions, we obtain that the CRC are also fulfilled for electrical intensity.

Thus, we have obtained that the electric and magnetic intensities of Maxwell's equations for a single-frequency quaternion in the form of a 4D vector satisfy the Cauchy-Riemann conditions.

5. Conclusion

Thus, using the quaternion multiplication operation and the Cauchy-Riemann conditions in vector representation, a quaternion vector is obtained, the elements of which include scalar and vector parts. The Cauchy-Riemann conditions correspond to the law of conservation of energy, therefore, the elements of the resulting vector must be equal to zero. It is shown that the equality of the scalar element to zero corresponds to Gauss's law for electrons and their spins. The vector elements of a quaternion create a circulation of intensities in vector space. Their equality to zero is due to the different direction of rotation of the left and right parts of the equations. Since all elements of the resulting quaternion vector are equal to zero, the solution to the vector equation will also be a quaternion.

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